大纲

- CFT basics.
  - classical / quantum CFT, Ward identities, 2/3-pt fns
    radial quant. and state/op correspondence, OPE
  - 2d CFT basics
  - Virasoro algebra and modules
  - VOA: chiral bosons, ghost,
    affine Kac-Moody (integrable, admissible, critical)
- SUSY and Class S theories
- localization and indices.
- SCFT/VOA correspondence
- Schur index and 2d qYM. (AGT 对应)

- Di Francesco, Mathieu, Senechal
- Alday, Conformal Field Theory
- Osborn and Petkos, hep-th/9307010

- Zhu, Modular invariance of characters of VOA

Mason, Tuite, Vertex op and modular form

crossing ratio

- 在 special conformal 变换下

$$|x_i' - x_j'| = \frac{|x_i - x_j|}{(1 - 2b\cdot x_i + b^2 x_i^2)(1 - 2b\cdot x_j + b^2 x_j^2)}$$

- 给定 4 个点 x_i, crossing ratio

$$\left|\frac{x_{12}\ x_{34}}{x_{13}\ x_{24}}\right| \qquad \left|\frac{x_{12}\ x_{34}}{x_{14}\ x_{23}}\right|$$

是 2 个独立的 conformal inv. 组合.

Summary of classical CFT

- Focus on flat space ($\mathbb{R}^n$, $g=\delta$)

- CF trans. (special change of local coord) $x^\mu \to x'^\mu$,

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Lambda(x)\, g_{\mu\nu}(x) \quad \text{some } \Lambda(x)$$

- 无穷小 CF: $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x)$

$$\Rightarrow \epsilon^\mu(x) = a^\mu \cdot 1 + A\delta^\mu{}_\nu x^\nu + \frac{1}{2}\underbrace{M_{\lambda\nu}}_{\text{反对称}}\left(g^{\mu\lambda}x^\nu - g^{\mu\nu}x^\lambda\right) + b^\sigma\left(g^\mu{}_\lambda g_{\sigma\nu} + g^\mu{}_\nu g_{\lambda\sigma} - g_{\nu\lambda} g^\mu{}_\sigma\right)x^\nu x^\lambda$$

将进入对应守恒流 $(j^a)^\mu = T^{\mu\nu}(\epsilon^a)_\nu$

- 考察对函数的作用

$$f(x) \longrightarrow f'(x') = f(x)$$

$$\Rightarrow \delta f(x) \equiv f'(x) - f(x)$$

$$\Rightarrow P_\mu = i\partial_\mu, \quad M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$$

$$D = -i x^\mu \partial_\mu \qquad K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2\partial_\mu) \Rightarrow \text{CFA}$$

Conformal Algebra

$$[D, P_\mu] = iP_\mu, \quad [D, K_\mu] = -iK_\mu, \quad [K_\mu, P_\nu] = 2i\left(\eta_{\mu\nu} D - L_{\mu\nu}\right)$$

$$[L_{\mu\nu}, P_\rho] = -i\left(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu\right), \quad [L_{\mu\nu}, K_\rho] = -i\left(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu\right)$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = -i\left(L_{\mu\rho}\eta_{\nu\sigma} - L_{\mu\sigma}\eta_{\nu\rho} - L_{\nu\rho}\eta_{\mu\sigma} + L_{\nu\sigma}\eta_{\mu\rho}\right)$$

$$[D, L_{\mu\nu}] = 0, \quad [P_\mu, P_\nu] = 0, \quad [K_\mu, K_\nu] = 0, \quad [D, D] = 0$$

From Alday's

- For $\mathbb{R}^{p,q}$: CFA $= so(p+1, q+1)$

  $= \mathbb{R}^{p+1, q+1}$ 的 some isometries

- fund. fields Φ forced to follow

$$\Phi(x) \longrightarrow \Phi'(x') = \mathcal{F}(\Phi, x)$$

例 $x \to x' = x + a$, scalar $\phi(x) \to \phi'(x') = \phi(x) = \mathcal{F}(\Phi, x)$

$x \to x' = \lambda x$, $\phi(x) \to \phi'(x') = \lambda^{-\Delta}\phi(x)$

- 经典共形不变性即为要求

$$S[\Phi] = \int d^n x\, \mathcal{L}(\Phi(x), \partial_\mu \Phi(x))$$

$\parallel$ classical conf. inv.

$$\int d^n x'\, \mathcal{L}(\Phi'(x'), \partial'_\mu \Phi'(x'))$$

$$\parallel$$

$$S[\Phi'] = \int d^n x\, \mathcal{L}(\Phi'(x), \partial_\mu \Phi'(x))$$

same coord.

• 例 $S = \int d^n x \; \partial_\mu \phi \partial^\mu \phi + \phi^k$

$$\Big\downarrow \quad \begin{aligned} & x \to x' = \lambda x \\ & \phi(x) \to \phi'(x') = \lambda^{-\Delta} \phi(x) \end{aligned}$$

$$\int d^n x' \; \partial'_\mu \phi'(x') \partial'^\mu \phi'(x') + \phi'(x')^k$$

$$= \int d^n x \; \lambda^n \left(\lambda^{-2\Delta} \frac{1}{\lambda^2} \partial_\mu \phi(x) \partial^\mu \phi(x) + \lambda^{-k\Delta} \phi(x) \right)$$

Scale inv. $\Rightarrow$ $\lambda^n \lambda^{-2\Delta} \lambda^{-2} = 1$ $\qquad$ $\lambda^n \lambda^{-k\Delta} = 1$

$\Rightarrow \Delta = \frac{n}{2} - 1$

$$\begin{aligned} k &= -\frac{n}{\Delta} \\ &= -\frac{n}{\frac{n}{2} - 1} \\ &= -\frac{2n}{n-2} \end{aligned}$$

Quantum CFT

- under $x \to x'$, $\Phi(x) \to \Phi'(x') = \mathcal{F}(\Phi, x)$,

  量子CFT要求 for any ops $\mathcal{O}_j$

$$\int D\Phi \prod_{j=1}^{n} \mathcal{O}_j(x_j') \, e^{-S[\Phi]} = \int D\Phi \prod_{j=1}^{n} \mathcal{O}_j'(x_j') \, e^{-S[\Phi]}$$

or

$$\left\langle \prod_{j=1}^{n} \mathcal{O}_j(x_j') \right\rangle = \left\langle \prod_{j=1}^{n} \mathcal{O}_j'(x_j') \right\rangle$$

- 定义 Primary 算符.
  (at the origin)
  用以控制 $\mathcal{O}'(x')$

$$[D, \mathcal{O}_\alpha(0)] = i\Delta\, \mathcal{O}_\alpha(0)$$

$$[M_{\mu\nu}, \mathcal{O}_\alpha(0)] = i(S_{\mu\nu})_\alpha{}^\beta\, \mathcal{O}_\beta(0)$$

$$[K_\mu, \mathcal{O}_\alpha(0)] = 0$$

$$\mathcal{O}_\alpha(x) = e^{-iP\cdot x}\, \mathcal{O}_\alpha(0)\, e^{+iP\cdot x}$$

$$\Rightarrow [\,CFA, \; \mathcal{O}_\alpha(x)\,]$$

$\Rightarrow$ $x \to x'$ $\qquad \mathcal{O}_I'(x') = \Lambda(x)^{-\frac{\Delta}{2}} (\text{Rotation})_I{}^J\, \mathcal{O}_J(x)$

时空指标

(Osborn and Petkos, eq (2.7))

where Λ is defined by $\quad g'_{\mu\nu}(x') = \Lambda(x)\, g_{\mu\nu}(x)$

Ward identity

- 在连续对称变换下

$$\delta_\epsilon \Phi(x) = \underbrace{\Phi'(x)}_{\text{同坐标}} - \Phi(x) = -i\epsilon^a G_a \Phi(x)$$

$$\delta_\epsilon \mathcal{O}(x_1,\cdots,x_N) = \mathcal{O}'(x_1,\cdots,x_N) - \mathcal{O}(x_1,\cdots,x_N)$$

$$\delta_\epsilon S[\Phi] = \int \partial_\mu j^\mu_a\, \epsilon^a(x)\, d^n x\ , \quad \partial_\mu j^\mu_a = 0 \text{ on shell}$$

守恒流

强行让变换参数依赖时空，由此导出 $\partial_\mu j^\mu$. $\epsilon^a(x)$ 是任意的

- "测度不变"假设 $\delta_\epsilon D\Phi = 0$

$\Rightarrow$ Ward identity $\delta_\epsilon \int D\Phi\ \mathcal{O}(x_1,\cdots,x_n)\, e^{-S[\Phi]} = 0$

$$\Updownarrow$$

$$\langle \delta_\epsilon \mathcal{O}(x_1,\cdots,x_N)\rangle - \int d^n x\ \epsilon_a(x) \langle \mathcal{O}(x_1,\cdots,x_N)\ \partial_\mu j^\mu_a(x)\rangle = 0$$

arbitrary, can of course chosen to be constant

• 若 $\mathcal{O}(x_1, \cdots, x_N) = \prod_{i=1}^{N} \mathcal{O}_i(x_i)$，则

$$\delta \mathcal{O}(x_1, \cdots, x_N)$$

$$= \sum_{i=1}^{N} \mathcal{O}_1(x_1) \cdots \underbrace{\delta \mathcal{O}_i(x_i)}_{-i\epsilon^a(x_i)\, G_a \mathcal{O}(x_i)} \cdots \mathcal{O}_N(x_N)$$

$$= -i \sum_{i=1}^{N} \int d^n x\, \delta(x - x_i)\, \epsilon^a(x)\, \mathcal{O}_1(x_1) \cdots G_a \mathcal{O}_i(x_i) \cdots \mathcal{O}_N(x_N)$$

$\downarrow$ ϵ 的任意性

$$\Rightarrow \underbrace{-i \sum_{i=1}^{N} \delta(x - x_i) \langle \mathcal{O}_1(x_1) \cdots G_a \mathcal{O}_i(x_i) \cdots \mathcal{O}_N(x_N) \rangle}_{\text{easy.}} = \underbrace{\langle \partial_\mu j_a^\mu(x) \prod_{i=1}^{N} \mathcal{O}_i(x_i) \rangle}_{\text{hard.}}$$

• 共形变换的流 ($j^\mu = T^\mu{}_\nu \epsilon^\nu$)

$T^\mu{}_\nu$	$T^{\mu[\nu} x^{\lambda]}$	$T^\mu{}_\nu x^\nu$	$T^\mu{}_\nu$(x 二次表达式)
(平移)	(转动)	(拉伸)	(SCF)

标量场关联函数与共形不变性

- 对于共形不变理论. 上述方程会极大约束 Primary 关联函数
  以标量理论为例.

$$F(\Phi, x) = \det\left(\frac{\partial x'}{\partial x}\right)^{-\frac{\Delta}{n}} \Phi(x)$$

$$\Rightarrow \quad \left\langle \prod_{i=1}^{N} \Phi(x_i') \right\rangle = \prod_{i=1}^{N} \det\left(\frac{\partial x'}{\partial x}\right)_{x=x_i} \left\langle \prod_{i=1}^{N} \Phi(x_i) \right\rangle$$

平移 $\quad x^\mu \to x'^\mu + a^\mu, \quad \det\frac{\partial x'}{\partial x} = 1$

$$\langle \Phi_1(x_1') \cdots \rangle = \langle \Phi_1(x_1) \cdots \rangle$$

Rotation $\quad x^\mu \to \Lambda^\mu{}_\nu x'^\nu, \quad \det = 1$

$$\langle \Phi_1(x_1') \cdots \rangle = \langle \Phi_1(x_1) \cdots \rangle$$

$\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = f(|x_1 - x_2|)$

all indices contracted

Dilatation $\quad x^\mu \to \lambda x^\mu, \quad \det = \lambda^n$

$$\langle \Phi_1(\lambda x_1) \cdots \rangle = \lambda^{-(\Delta_1 + \cdots)} \langle \Phi_1(x_1) \cdots \rangle$$

$\Rightarrow$ 2-pt $\quad \langle \Phi_1(x_1)\Phi_2(x_2)\rangle = f(|x_1 - x_2|)$ 是齐次函数

$$\Rightarrow \quad \langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}}$$

Special conformal inv.

$$x^\mu \to x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b\cdot x + b^2 x^2} \qquad \det \frac{\partial x'}{\partial x} = \frac{1}{(1-2b\cdot x + b^2 x^2)^n}$$

$$\Rightarrow \langle \Phi_1(x_1') \Phi_2(x_2') \rangle = (1-2b\cdot x_1 + b^2 x_1^2)^{\Delta_1} (1-2b\cdot x_2 + b^2 x_2^2)^{\Delta_2} \langle \Phi_1(x_1) \Phi_2(x_2) \rangle$$

$$\frac{C_{12}}{|x_1' - x_2'|^{\Delta_1+\Delta_2}} \qquad = \qquad \frac{(1-2b\cdot x_1 + b^2 x_1^2)^{\Delta_1} (1-2b\cdot x_2 + b^2 x_2^2)^{\Delta_2} C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

$$= \frac{[(1-2b\cdot x_1 + b^2 x_1^2)(1-2b\cdot x_2 + b^2 x_2^2)]^{\frac{\Delta_1+\Delta_2}{2}}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} C_{12} \qquad //$$

$$\Rightarrow C_{12} = 0 \quad \text{if } \Delta_1 \neq \Delta_2$$

- 三点关联函数

$$\langle \Phi_1(x_1) \Phi_2(x_2) \Phi(x_3) \rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} \, x_{23}^{\Delta_2+\Delta_3-\Delta_1} \, x_{13}^{\Delta_3+\Delta_1-\Delta_2}}$$

$$= \frac{C_{123}}{\prod_{i<j} x_{ij}^{\Delta_i+\Delta_j-\Delta_k}}$$

$$\langle \prod_{i=1}^{4} \Phi_i(x_i) \rangle = f\left(\frac{x_{12}\, x_{34}}{x_{13}\, x_{24}}, \frac{x_{12}\, x_{34}}{x_{23}\, x_{14}} \right) \prod_{i<j} \frac{1}{x_{ij}^{\Delta_i+\Delta_j - \frac{1}{3}\Delta}}$$

共形不变性与张量关联函数

· 例 矢量场 J_μ

① 平移不变性，scaling inv, Lorentz 不变性. μ,ν 对称

$$\Rightarrow \langle J_\mu(x) J_\nu(y)\rangle = \frac{\alpha_{\mu\nu}(x-y)}{|x-y|^{2\Delta}} \quad , \quad \alpha_{\mu\nu}(x) = \delta_{\mu\nu} + \alpha \frac{x_\mu x_\nu}{x^2}$$

② 特殊共形变换不变性：固定

$$\alpha_{\mu\nu} = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \equiv I_{\mu\nu}$$

③ 若 J_μ 是量子守恒流，即 $\partial_\mu J^\mu = 0$ as op 方程 则

$$\partial^\mu \langle J_\mu(x) J_\nu(y)\rangle = 0 \Rightarrow \partial^\mu \frac{I_{\mu\nu}(x-y)}{|x-y|^{2\Delta}} = 0$$

$$\Rightarrow \frac{-2\frac{(\delta^\mu_\mu x_\nu + x_\mu \delta^\mu{}_\nu)x^2 - 2x^\mu x_\mu x_\nu}{x^4}}{|x|^{2\Delta}} + \left(\delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\right)\frac{-\Delta\, 2x^\mu}{|x|^{2\Delta+2}} = 0$$

$$-2\frac{n x_\nu - x_\nu}{|x|^{2\Delta+2}} + \frac{(-x_\nu)}{|x|^{2\Delta+2}}(-2\Delta) = 0$$

$$\Rightarrow \quad \Delta = n-1$$

• 例：2阶张量场

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle$$

$$= \frac{1}{|x-y|^{2\Delta}}\left(\frac{1}{2} I_{\mu\sigma}(x-y) I_{\nu\rho}(x-y) + I_{\mu\rho}(x-y) I_{\nu\sigma}(x-y) - \frac{1}{n}\delta_{\mu\nu}\delta_{\rho\sigma}\right)$$

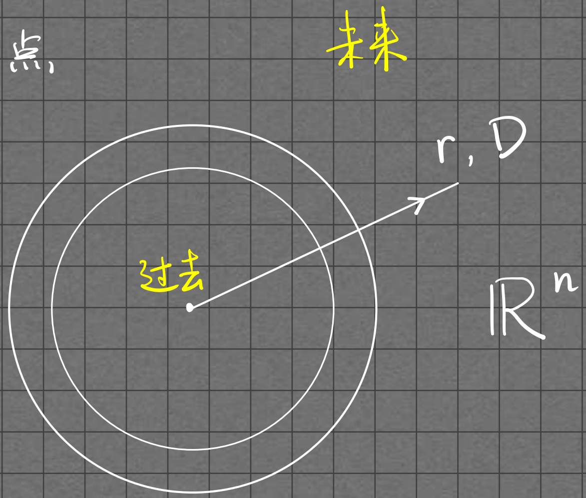
且 $\partial^\mu T_{\mu\nu} = 0$ 同样给出 $\Delta = n$

• 一般 2点，3点，见 Osborn, Petkos, eq (2.8) eq (2.13)

Radial quantization

- $\mathbb{R}^n$ 没有时间方向：重新定义"等时面"为同心球面.
- 无穷远过去：原点

  无穷远未来：无穷远点.



- Dilatation D 扮演哈密顿量角色

  $\Rightarrow$ 本征态作完备基 $\{|\Delta\rangle\}$

- Radial ordering $\quad R\, \mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \begin{cases} \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2) & |x_1| > |x_2| \\ \mathcal{O}_2(x_2)\,\mathcal{O}_1(x_1) & |x_2| > |x_1| \end{cases}$

- 定义 $|0\rangle \in$ Hilbert 为"唯一共形不变真空态", s.t.

$$D|0\rangle = P|0\rangle = K|0\rangle = M|0\rangle = 0$$

$\Rightarrow$ 对 primary 算符 $\mathcal{O}$, $|\mathcal{O}\rangle \equiv \mathcal{O}(0)|0\rangle$ 满足

$$D|\mathcal{O}\rangle = [D, \mathcal{O}(0)]|0\rangle = i\Delta|\mathcal{O}\rangle$$

$$K|\mathcal{O}\rangle = [K, \mathcal{O}(0)]|0\rangle = 0$$

$\Rightarrow$ 定义: $|\mathcal{O}\rangle$ 的 descendant states $P^n|\mathcal{O}\rangle$ ($KP^n|\mathcal{O}\rangle \neq 0$)

是 D 的本征态, 本征值 $i(\Delta+n)$

对 $\mathcal{O}(x) \equiv e^{iP\cdot x}\mathcal{O}(0)e^{-iP\cdot x}$, $\mathcal{O}(x)|0\rangle$ 则不是 D 的本征态

$$\mathcal{O}(x)|0\rangle = e^{iPx}\mathcal{O}(0)e^{-iPx}|0\rangle = e^{iPx}|\mathcal{O}\rangle$$

高维Taylor expansion

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} x^n \underbrace{\partial^n \mathcal{O}(0)}_{\text{descendant operators.}}|0\rangle$$

$\Rightarrow$ $\mathcal{O}(x)|0\rangle$ 不是 D 的本征态, 是 $|\mathcal{O}\rangle$ 与其 descendants 的叠加.

- "State / op correspondence":

any state 1-1 corresponds to an op inserted at the origin.

$$\mathcal{O} \longrightarrow |\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle$$

$$|\mathcal{O}\rangle \longrightarrow \mathcal{O} \quad (\text{valid for conformal theory})$$

OPE

- 设 x 在 0 附近的地方.

$$\mathcal{O}_1(x)\,\mathcal{O}_2(0)\,|0\rangle = \sum_{\Delta} C_\Delta(x)\,|\Delta\rangle$$ (用 D 的本征态 $|\Delta\rangle$ 展开)

$\downarrow$ state/op 对应

$$= \sum_{\substack{\text{primary}\\ \mathcal{O}}} \underbrace{C_{12}^{\mathcal{O}}(x,\partial)\,\mathcal{O}(0)}_{\mathcal{O}(0)\text{ 与 } \partial^n\mathcal{O}(0)\text{ 的复杂组合}}\,|0\rangle$$

- 移除 $|0\rangle$ $\Rightarrow$ OPE (operator product expansion)

$$\mathcal{O}_1(x)\,\mathcal{O}_2(0) = \sum_{\text{primaries}} C_{\mathcal{O}}^{12}(x,\partial)\,\mathcal{O}(0)$$ (valid in correlators $\langle 0|\cdots|0\rangle$)

$$= \left(\underbrace{\frac{C_{12\mathcal{O}}}{|x|^k}\,\mathcal{O}_\Delta(0)}_{\substack{\text{leading term}\\ \text{some } k \text{ to}\\ \text{be fixed}}} + \underbrace{\cdots\ \text{descendants}}_{\text{subleading}}\right) + \left(\text{其它 primaries 及 descendants}\right)$$

- 实际上 $C_{12}^{\mathcal{O}}(x,\partial)$ 是由 CF 不变性完全确定

(up to 常数因子 $C_{12\mathcal{O}}$)

- $$D\left(\mathcal{O}_1(x)\,\mathcal{O}_2(0)|0\rangle\right) = [D,\mathcal{O}_1(x)]\,\mathcal{O}_2(0)|0\rangle + \mathcal{O}_1(x)\,[D,\mathcal{O}_2(0)]\,|0\rangle$$

$$= i(\Delta_1 + x^\mu\partial_\mu)\,\mathcal{O}_1(x)\,\mathcal{O}_2(0)|0\rangle + i\Delta_2\,\mathcal{O}_1(x)\,\mathcal{O}_2(0)|0\rangle$$

$$= \sum_{\substack{\text{primary}\\ \mathcal{O}}}\left[i(\Delta_1+\Delta_2) + i x^\mu\partial_\mu\right]\frac{C_{12\mathcal{O}}}{|x|^k}\,\mathcal{O}_\Delta(0)\,|0\rangle$$

$$= \sum_{\substack{\text{primary}\\ \mathcal{O}}} i\,\underbrace{(\Delta_1+\Delta_2-k)}\,\frac{C_{12\mathcal{O}}}{|x|^k}\,\mathcal{O}_\Delta(0)\,|0\rangle$$

$$\Rightarrow \quad \Delta = \Delta_1 + \Delta_2 - k$$

$$D\sum_{\substack{\text{primary}\\ \mathcal{O}}}\frac{C_{12\mathcal{O}}}{|x|^k}\,\mathcal{O}_\Delta(0)|0\rangle = \sum_{\substack{\text{primary}\\ \mathcal{O}}} \underbrace{i\Delta}\,\frac{C_{12\mathcal{O}}}{|x|^k}\,\mathcal{O}_\Delta(0)|0\rangle$$

$\Rightarrow$ leading term is fixed.

- Scalar O_1, O_2:

$$O_1(x)\,O_2(0) \overset{\text{scale inv}}{=} \frac{1}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(O_\Delta(0) + \alpha\, x^\mu \partial_\mu O_\Delta(0) + \cdots\right)$$

两边作用 K_μ 做比较

$$K_\mu O_1(x) O_2(0)|0\rangle = [K_\mu, O_1(x)]\, O_2(0)|0\rangle + O_1(x)\overset{=0}{[K_\mu, O_2(0)]}|0\rangle$$

$$= \left[2i\, x_\mu \Delta_1 O_1(x) + i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) O_1(x)\right] O_2(0)|0\rangle$$

$$= \left[2i\, x_\mu \Delta_1 + i(2x_\mu x^\nu \partial_\nu - x^2\partial_\mu)\right] \frac{1}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(O_\Delta(0) + \alpha\, x^\mu \partial_\mu O_\Delta(0) + \cdots\right)|0\rangle$$

$$= \frac{i\, x_\mu(\Delta+\Delta_1-\Delta_2)}{|x|^{\Delta_1+\Delta_2-\Delta}} O_\Delta(0)|0\rangle + \cdots$$

而

$$K_\mu\left[\frac{1}{|x|^{\Delta_1+\Delta_2-\Delta}}\left(O_\Delta(0)|0\rangle + \alpha\, x^\mu\partial_\mu O_\Delta(0)|0\rangle + \cdots\right)\right]$$

$$= \frac{-i}{|x|^{\Delta_1+\Delta_2-\Delta}}\,\alpha\, x^\nu K_\mu\, i\partial_\nu O_\Delta(0)|0\rangle$$

$$= \frac{-i}{|x|^{\Delta_1+\Delta_2-\Delta}}\,\alpha\, x^\nu K_\mu P_\nu O_\Delta(0)|0\rangle$$

$$= \frac{-i}{|x|^{\Delta_1+\Delta_2-\Delta}}\,\alpha\, x^\nu\, 2i(\delta_{\mu\nu} D - M)\, O_\Delta(0)|0\rangle$$

$$= \frac{2\alpha}{|x|^{\Delta_1+\Delta_2-\Delta}}\, x_\mu \Delta\, O_\Delta(0)|0\rangle$$

$$\Rightarrow \frac{2\alpha}{|x|^{\Delta_1+\Delta_2-\Delta}}\, x_\mu \Delta = \frac{i\, x_\mu(\Delta+\Delta_1-\Delta_2)}{|x|^{\Delta_1+\Delta_2-\Delta}}$$

$$\Rightarrow \alpha = \frac{\Delta+\Delta_1-\Delta_2}{2\Delta}$$

即 sub leading term 也被 fixed.

- 实际上 $C_{\mathcal{O}}(x,\partial)$ 是由 CF 不变性完全确定 (up to C_{12A} 常数)

- Scalar ops.

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\rangle = \sum_{\mathcal{O}'} C_{12\mathcal{O}'}\,C_{\mathcal{O}'}(x_1-x_2,\partial)\,\langle \mathcal{O}'(x_2)\,\mathcal{O}_3(x_3)\rangle$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$C_{123}\,C_3(x_1-x_2,\partial)\,\langle \mathcal{O}_3(x_2)\,\mathcal{O}_3(x_3)\rangle$$

$$\frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\;x_{13}^{\Delta_1+\Delta_3-\Delta_2}\;x_{23}^{\Delta_2+\Delta_3-\Delta_1}} \qquad\qquad C_{123}\,C_3(x_1-x_2,\partial)\,\frac{1}{x_{23}^{2\Delta_3}}$$

两边对比可得 $C_3^{12}(x_1-x_2,\partial)$ 的表达式

2d CFT

- $\exists$ local CF : $\forall\ z \to z' = f(z)$

$$g = g_{z\bar z}\, dz\, d\bar z \to g'_{z\bar z}\, dz' d\bar z' = g'_{z\bar z} \frac{df}{dz}\frac{d\bar f}{d\bar z} dz\, d\bar z$$

$$\Rightarrow \quad g'_{z\bar z}(z', \bar z') = g_{z\bar z}(z,\bar z) \underbrace{\left(\frac{df}{dz}\right)^{-1}\left(\frac{d\bar f}{d\bar z}\right)^{-1}}_{\Lambda(z,\bar z)}$$

- 无穷小变换 $\quad z \to z' = z + \epsilon(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n z^n$

- $z \to z' = z + \underset{\mathbb{C}}{\underset{\uparrow}{\epsilon_0}} + \underset{\mathbb{C}}{\underset{\uparrow}{\epsilon_1}}\, z + \underset{\mathbb{C}}{\underset{\uparrow}{\epsilon_2}}\, z^2$ 对应平移、转动，拉伸

SCF.

- Ward identities

$$\delta_\epsilon \left\langle \prod_{j=1}^{n} \mathcal{O}_j(x_j) \right\rangle = \int d^2x \left\langle \prod_{j=1}^{n} \mathcal{O}_j(x_j)\, \partial_\mu \left(T^{\mu\nu}\epsilon_\nu\right) \right\rangle$$

$$\sim -\oint_{大C} \left\langle \prod_{j=1}^{n} \mathcal{O}_j(x_j) \left(T_{zz}\,\epsilon^{z}(z,\bar z)\,dz - T_{\bar z\bar z}\,\epsilon^{\bar z}(z,\bar z)\,d\bar z \right) \right\rangle$$

$$\boxed{T(z) \equiv -2\pi T_{zz}}$$

$$+ \cancelto{0}{T_{\bar z z}\,\epsilon^{z}(z,\bar z)}\,\frac{dz}{2\pi i} - \cancelto{0}{T_{z\bar z}\,\epsilon^{\bar z}(z,\bar z)}\,\frac{d\bar z}{2\pi i}) \rangle$$

（来自 Ward id $\langle T_{z\bar z}\,\mathcal{O}_j(z_j)\rangle \sim \delta(z-z_j)$

大C 积分远离所有 z_j ）

相当于等时面积分.

- In terms of "charge" $\quad Q = -\dfrac{1}{2\pi i}\oint dz\, T(z)\,\epsilon(z) + d\bar z\,\bar T(\bar z)\,\bar\epsilon(\bar z)$

$$\Rightarrow \quad \delta_\epsilon \mathcal{O}(z,\bar z) = [Q, \mathcal{O}(z,\bar z)] = -\oint_z \frac{dw}{2\pi i}\, T(w)\,\epsilon(w)\,\mathcal{O}(z,\bar z)$$

$$\delta_{\bar\epsilon} \mathcal{O}(z,\bar z) = [Q, \mathcal{O}(z,\bar z)] = -\oint_z \frac{dw}{2\pi i}\, \bar T(w)\,\bar\epsilon(\bar w)\,\mathcal{O}(z,\bar z)$$

2d primary ops

- 定义 primary：

$$\overbrace{z \to z' = f(z) \qquad \bar{z} \to \bar{z}' = \tilde{f}(\bar{z})}^{\text{独立}}$$

$$\mathcal{O}'(z', \bar{z}') = \left(\frac{dz'}{dz}\right)^{-h} \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{-\bar{h}} \mathcal{O}(z, \bar{z})$$

- 无穷小、 $z \to z' = z + \epsilon^z(z)$, $\bar{z} \to \bar{z}' = \bar{z} + \bar{\epsilon}^{\bar{z}}(\bar{z})$

$$\delta \mathcal{O}(z, \bar{z}) = \mathcal{O}'(z, z) - \mathcal{O}(z, \bar{z}) = -(h \mathcal{O} \partial_z \epsilon + \epsilon \partial_z \mathcal{O}) - (\bar{h} \mathcal{O} \partial_{\bar{z}} \bar{\epsilon} + \bar{\epsilon} \partial_{\bar{z}} \mathcal{O})$$

- under $\forall \epsilon^z(z, \bar{z}) = \epsilon^z(z)$ $\qquad \bar{\epsilon}^{\bar{z}} = 0$, Ward identity

$$\sum_{j=1}^{n} \left(\epsilon^z(z_j) \partial_{z_j} + \partial \epsilon^z(z_j) h_j\right) \left\langle \Pi_{j=1}^{n} \mathcal{O}_j(z_j, \bar{z}_j) \right\rangle$$

$$= \oint_{\text{大}C} \frac{dz}{2\pi i} \left\langle \Pi_{j=1}^{n} \mathcal{O}_j(x_j) \, T(z) \, \epsilon^z(z) \right\rangle$$

- When $n=1$. 可推得 OPE for primary

$$T(z) \, \mathcal{O}(w, \bar{w}) \sim \frac{h \, \mathcal{O}(w)}{(z-w)^2} + \frac{\partial \mathcal{O}(w)}{z-w}$$

$$\bar{T}(\bar{z}) \, \mathcal{O}(w, \bar{w}) \sim \frac{\bar{h} \, \mathcal{O}(w, \bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial} \mathcal{O}(w, \bar{w})}{\bar{z}-\bar{w}}$$

- Note: $T, \bar{T}$ is not primary.

primary two-pt / three-pt (w.r.t. unique vacuum)

- $\langle \mathcal{O}_1(z,\bar{z})\, \mathcal{O}_2(w,\bar{w})\rangle = \dfrac{C_{12}}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}}$ if $h_1 = h_2 = h$ $\bar{h}_1 = \bar{h}_2 = \bar{h}$

- $\langle \mathcal{O}_1(z_1,\bar{z}_1)\, \mathcal{O}_2(z_2,\bar{z}_2)\, \mathcal{O}_3(z_3,\bar{z}_3)\rangle$

$$= \frac{C_{123}}{z_{12}^{h_1+h_2-h_3}\, z_{13}^{h_1+h_3-h_2}\, z_{23}^{h_2+h_3-h_1}} \cdot (\text{c.c.})$$

- Note that these formular work for spinful ops ($h_i - \bar{h}_i \neq 0$)

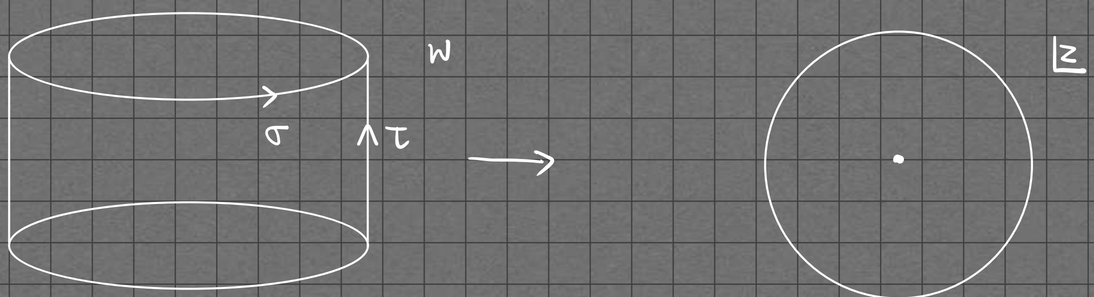
实标量场

- $S = \int_M \sqrt{g}\, d^2x\, \partial_\mu X \partial^\mu X$

- 考虑 $\mathbb{R}^1_t \times S^1_\sigma$, $w = \underset{t=-i\tau}{\tau} + i\sigma = i(t+\sigma)$, $\bar{w} = \tau - i\sigma = i(t-\sigma)$

$$S \propto \int_{-\infty}^{+\infty} dt \int_0^{2\pi} d\sigma \ (\partial_t X)^2 - (\partial_\sigma X)^2$$

$$S_E \propto \int dw\, d\bar{w} \ \ \partial_w X\, \partial_{\bar{w}} X$$



- 在 坐标变换 (conformal) $w \to z = z(w)$, $\bar{w} \to \bar{z} = \bar{z}(\bar{w})$,
$X(w,\bar{w}) \to X'(z(w), \bar{z}(\bar{w})) = X(w,\bar{w})$ 下

$$S \to \int dz\, d\bar{z} \ \ \partial_z X' \partial_{\bar{z}} X'$$

$$= \int dw\, d\bar{w} \frac{dz}{dw} \frac{d\bar{z}}{d\bar{w}} \frac{dw}{dz} \frac{d\bar{w}}{d\bar{z}} \qquad \partial_w X\, \partial_{\bar{w}} X$$

$$= \int dw\, d\bar{w} \ \partial_w X\, \partial_{\bar{w}} X$$

即 S_E 共形不变

- 守恒流: $T^{\mu}{}_{\nu} \sim \left(\partial^{\mu} X \partial_{\nu} X - \frac{1}{2} \partial_{\lambda} X \partial^{\lambda} X \, \delta^{\mu}{}_{\nu}\right)$

- mode expansion

$$X(t,\sigma) = \underbrace{x_0 + p_0 t}_{\text{质心位置.动量}} + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n} a_n^{\pm} e^{-in(t\pm\sigma)}$$

- 对易关系

$$[x_0, p_0] = i \qquad [\alpha_m^{\pm}, \alpha_n^{\pm}] = m\,\delta_{m+n,0}$$

$$\Leftrightarrow \quad [X(t,\sigma), \pi(t,\sigma')] = +i\,\delta(\sigma-\sigma')$$

其中 $\pi(t,\sigma) = \frac{1}{2\pi} \partial_t X(t,\sigma)$

- $X(t,\sigma)^{\dagger} = X(t,\sigma) \Rightarrow (\alpha_m^{\pm})^{\dagger} = \alpha_{-m}^{\pm}$

约定 $\alpha^{\pm}_{m>0}$ 是"湮灭", $\alpha^{\pm}_{m<0}$ 是"生成"

- Normal ordering

$$:\alpha_{-m}\alpha_m: = \alpha_{-m}\alpha_m \qquad , \forall m>0 \qquad :x_0 p_0: = :p_0 x_0: = x_0 p_0$$
$$:\alpha_m \alpha_{-m}: = \alpha_{-m}\alpha_m$$

- OPE on $S^1_\sigma \times \mathbb{R}^1_t$

$$X(t_1,\sigma_1)\,X(t_2,\sigma_2)$$

$$= :X(t_1,\sigma_1)\,X(t_2,\sigma_2): - it_1 + \frac{1}{2}\sum_{\pm}\sum_{m>0}\frac{1}{m}\, e^{-im\left[(t_1\pm\sigma_1)-(t_2\pm\sigma_2)\right]}$$

Note $$\frac{i^2}{2}\sum_{\substack{m>0\\ n<0}}\frac{1}{m}\,\frac{1}{n}\,\overbrace{\alpha_m^{\pm}\,\alpha_n^{\pm}}^{\text{not yet normal ordered}}\, e^{-im(t_1\pm\sigma_1)-in(t_2\pm\sigma_2)}$$

$$= -\frac{1}{2} :\sum_{\substack{m>0\\ n<0}}(\cdots): - \frac{1}{2}\sum_{\pm}\sum_{\substack{m>0\\ n<0}}\frac{1}{mn}\, m\,\delta_{m+n,0}\, e^{-im(t_1\pm\sigma_1)-in(t_2\pm\sigma_2)}$$

$$= :\cdots: - \frac{1}{2}\sum_{\pm}\sum_{m>0}\frac{1}{-m}\, e^{-im\left[(t_1-t_1)\pm(\sigma_1\pm\sigma_2)\right]}$$

$$= :\cdots: + \frac{1}{2}\sum_{\pm}\sum_{m>0}\frac{1}{m}\, e^{-im\left[(t_1-t_1)\pm(\sigma_1\pm\sigma_2)\right]}$$

- 求和不收斂 (when $(t_1\pm\sigma_1)-(t_2\pm\sigma_2)=0$)

$$\sum_{m>0}\frac{1}{m} \to \infty$$

- Wick rotation $t \equiv -i\tau, \quad \tau \in \mathbb{R}$

记 $z = e^{\underbrace{\tau + i\sigma}_{w}}, \quad \bar{z} = e^{\underbrace{\tau - i\sigma}_{\bar{w}}}$

$$X(\tau_1,\sigma_1)\, X(\tau_2,\sigma_2)$$

$$= :X(\tau_1,\sigma_1)\, X(\tau_2,\sigma_2): - \tau_1 + \underbrace{\frac{1}{2}\sum_{m>0}\frac{1}{m}\left(\frac{z_2}{z_1}\right)^m + \frac{1}{2}\sum_{m>0}\frac{1}{m}\overline{\left(\frac{z_2}{z_1}\right)^m}}$$

when $|z_2| < |z_1| \Leftrightarrow \tau_2 < \tau_1$ 收敛

$$X(\tau_1,\sigma_2)\, X(\tau_2,\sigma_2) = :X(\tau_1,\sigma_2)\, X(\tau_2,\sigma_2): - \log|z_1 - z_2|$$

提 $\mathbb{C}$ 上 OPE

$$R\, X(z_1)\, X(z_2) = :X(z_1)\, X(z_2): - \log|z_1 - z_2|$$

$$= R\, X(z_2)\, X(z_1)$$

• on $\mathbb{C}$, $z = e^{w}$ $\bar{z} = e^{\bar{w}}$

• $X(z,\bar{z}) = x_0 - \frac{i}{2} p_0 \ln z\bar{z} + \frac{i}{\sqrt{2}} \sum_n{}' \frac{1}{n} \alpha_n^{+} z^{-n} + \frac{i}{\sqrt{2}} \sum_n{}' \frac{1}{n} \alpha_n^{-} \bar{z}^{-n}$

• 常用 convention rescale $X \to \sqrt{2} X$ s.t.

$$R\, X(z_1,\bar{z}_1)\, X(z_2,\bar{z}_2) = :X(z_1,\bar{z}_1)\, X(z_2,\bar{z}_2): - \log z_1 - z_2 - \log(\bar{z}_1 - \bar{z}_2)$$

• $$R\, \partial X(z,\bar{z})\, \partial X(w,\bar{w}) = -\frac{1}{(z-w)^2} + :\partial X(z)\, \partial X(w):$$

$$= -\frac{1}{(z-w)^2} + \text{reg}\,(\text{as } z \to w)$$

省略不写 $\bar{z}$

取 $z \to w$

• $$T(z) = -\frac{1}{2} :\partial X(z)\, \partial X(z):$$

$$= -\frac{1}{2} \lim_{z \to w} R\, \partial X(z) \partial X(w) - \langle \partial X(z) \partial X(w) \rangle$$

("point-splitting" def of $T(z)$)

• 暴力算

$$R\, T(z)\, \partial X(w) = -\frac{1}{2} :\partial X(z)\, \partial X(z)\, \partial X(w): - \frac{1}{2} \cdot 2 \left[-\frac{1}{(z-w)^2} \right] \partial X(z)$$

$$= \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w}$$

$$\underbrace{- \frac{1}{2} :\partial X(z)\, \partial X(z)\, \partial X(w): + \sum_{n=2} \frac{1}{n!} \frac{\partial^{n+1} X(w)}{(z-w)^2} (z-w)^n}_{\text{Reg. as } z \to w}$$

• $$R\, T(z)\, T(w) = \frac{1}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + O(z-w)$$

- Vertex operator $V_\alpha(z) \equiv :e^{i\sqrt{2}\alpha X}:$

$$\Rightarrow \quad T(z)\, V_\alpha(w) = \alpha^2 \frac{V_\alpha(w)}{(z-w)^2} + \frac{\partial_w V_\alpha(w)}{z-w} + O(z-w)$$

$$\partial X(z)\, V_\alpha(w) = -i\alpha \frac{V_\alpha(w)}{z-w} + O(z-w)$$

图利

$$\begin{cases} e^A e^B = e^B e^A e^{[A,B]} , \quad \text{when } [A,B] \text{ commutes with } A,B \\ e^{wa} e^{za^\dagger} = e^{za^\dagger} e^{wa} e^{wz[a,a^\dagger]} \\ [a, e^{za^\dagger}] = z e^{za^\dagger} \end{cases}$$

- $$\partial X(z) = -\frac{i}{2}\sqrt{2}\, p_0 \frac{1}{z} - i \sum_n{}' \alpha_n^+ z^{-n-1}$$

$$= i \sum_n a_n z^{-n-1} \qquad \Rightarrow \qquad [a_m, a_n] = m\,\delta_{m+n,0}, \quad m,n \neq 0$$

$$T(z) = -\frac{1}{2} :\partial X \partial X:(z)$$

$$= +\frac{1}{2} \sum_{m,n} :a_n a_m: \; z^{-n-1} z^{-m-1}$$

$$= \sum_N \sum_n \frac{1}{2} :a_n a_{N-n}: \; z^{-N-1} = \sum_N L_N z^{-N-1}$$

$$L_{N\neq 0} = \frac{1}{2} \sum_n :a_n a_{N-n}:$$

$$L_0 = \frac{1}{2} \sum_n :a_n a_{-n}: = +\frac{1}{2} a_0^2 + \sum_{n>0} a_{-n} a_n$$

$$\Rightarrow \quad [L_0, a_{m\neq 0}] = \Big[\sum_{n>0} a_{-n} a_n , a_m\Big] = m\, a_m$$