- Vertex operator $V_\alpha(z) \equiv \, :e^{i\sqrt{2}\alpha X}:$

$$\Rightarrow \quad T(z)\, V_\alpha(w) = \alpha^2 \frac{V_\alpha(w)}{(z-w)^2} + \frac{\partial_w V_\alpha(w)}{z-w} + O(z-w)$$

$$\partial X(z)\, V_\alpha(w) = -i\alpha \frac{V_\alpha(w)}{z-w} + O(z-w)$$

用到

$$\begin{cases} e^A e^B = e^B e^A e^{[A,B]}, & \text{when } [A,B] \text{ commutes with } A, B \\ e^{wa} e^{za^\dagger} = e^{za^\dagger} e^{wa} e^{wz[a,a^\dagger]} \\ [a, e^{za^\dagger}] = z e^{za^\dagger} \end{cases}$$

- $$\partial X(z) = -\frac{i}{2}\sqrt{2}\, p_0 \frac{1}{z} - i \sum_n{}' \alpha_n^+ z^{-n-1}$$

$$= i \sum_n a_n z^{-n-1} \qquad \Rightarrow \qquad [a_m, a_n] = m\,\delta_{m+n,0}, \quad m,n \neq 0$$

$$T(z) = -\frac{1}{2} :\partial X \partial X:(z)$$

$$= +\frac{1}{2} \sum_{m,n} :a_n a_m: \; z^{-n-1} z^{-m-1}$$

$$= \sum_N \sum_n \frac{1}{2} :a_n a_{N-n}: \; z^{-N-1} = \sum_N L_N z^{-N-1}$$

$$L_{N\neq 0} = \frac{1}{2} \sum_n :a_n a_{N-n}:$$

$$L_0 = \frac{1}{2} \sum_n :a_n a_{-n}: = +\frac{1}{2} a_0^2 + \sum_{n>0} a_{-n} a_n$$

$$\Rightarrow \quad [L_0, a_{m\neq 0}] = \Big[\sum_{n>0} a_{-n} a_n, a_m\Big] = m\, a_m$$

Real (Majorana) fermion

- 在 $\mathbb{R}\times S^1$ 上做量子化 ($w=\tau+i\sigma$, $w\sim w+2\pi$)

$$S=\int dt\,d\sigma\ \Psi^\dagger\gamma^0\gamma^\mu\partial_\mu\Psi$$

$$\xrightarrow[w=\tau+i\sigma]{\text{Wick}}\int dw\,d\bar{w}\quad \psi\,\partial_{\bar{w}}\psi+\tilde{\psi}\,\partial_w\tilde{\psi}$$

$$\left.\begin{array}{lcc} h & \frac{1}{2} & 0 \\ \bar{h} & 0 & \frac{1}{2} \end{array}\right\}\text{才能共形不变}$$

$$\gamma^t=\begin{pmatrix}0&i\\i&0\end{pmatrix}\quad \gamma^\sigma=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad C_+=\begin{pmatrix}0&1\\1&0\end{pmatrix}\qquad \Psi=\begin{pmatrix}\psi\\\tilde{\psi}\end{pmatrix}$$

- EOM: $\partial_{\bar{w}}\psi=0\qquad \partial_w\tilde{\psi}=0$

- Ramond sector (PB): $\psi(w)=\sum_{n\in\mathbb{Z}} b_n\,\underbrace{e^{-nw}}$ 周期性

  "zero mode" $= \underbrace{b_0} +$ the rest

  Neveu-Schwarz (APB): $\psi(w)=\sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n\,\underbrace{e^{-nw}}$ 反周期性

- 量子化.

  (R or NS) $\quad\{\psi(\tau,\sigma_1),\psi(\tau,\sigma_2)\}=\delta(\sigma_1-\sigma_2)$

  $$\Leftrightarrow\ \{b_m,b_n\}=\delta_{m+n}$$

  Ramond: including $\{b_0,b_0\}=1$, b_0 invertible

- Normal ordering: 按下标 从小到大排.

- $\underline{\underline{T_{cyl}}} = - :\psi \partial_w \psi:$ $\qquad \underline{\underline{\mathbb{R}^1 \times S^1}} \vdash T$

$$= \sum_{m,n} n :b_m b_n: \; e^{-(m+n)w} \quad + \text{ambiguity}$$

$$= \sum_{N} \sum_{n} n :b_{N-n} b_n: \; e^{-Nw} \quad + \text{ambiguity}$$

• Mapping to plane. , $z = e^w$

$$\psi(w) \to \underset{\substack{\uparrow \\ \text{plane}}}{\psi_p(z)} = \left(\frac{dz}{dw}\right)^{\overset{-h}{-\frac{1}{2}}} \psi(w)$$

Ramond cyl :
$$\psi_p(z) = z^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}} b_n z^{-n} = \sum_{n\in\mathbb{Z}} b_n z^{\underbrace{-n-\frac{1}{2}}_{\text{半整}}} : R_p$$

$$\psi_p(z \to e^{2\pi i} z) \to -\psi_p(z)$$

NS cyl :
$$\psi_p(z) = z^{-\frac{1}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n z^{-n} = \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n z^{\underbrace{-n-\frac{1}{2}}_{\text{整}}} : NS_p$$

$$\psi_p(z \to e^{2\pi i} z) = \psi_p(z)$$

• NS sector ($\mathbb{C}$上周期)

$$R\,\psi_p(z_1)\psi_p(z_2) = z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} \sum_{m,n\in\mathbb{Z}+\frac{1}{2}} b_m b_n z_1^{-m} z_2^{-n}$$

已天然 normal ordered

$$= \cdots + z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} \sum_{m=\frac{1}{2}}^{+\infty+\frac{1}{2}} b_m b_{-m} z_1^{-m} z_2^{m}$$

$$= :\psi_p(z_1)\psi_p(z_2): + z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} \left(\frac{z_2}{z_1}\right)^{\frac{1}{2}} \sum_{m=0}^{+\infty} \left(\frac{z_2}{z_1}\right)^{m}$$

$$= :\psi_p(z_1)\psi_p(z_2): + \frac{1}{z_1 - z_2}$$

• R - Sector

$$R\,\psi_p(z_1)\psi_p(z_2) = z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} \sum_{m,n\in\mathbb{Z}} b_m b_n z_1^{-m} z_2^{-n}$$

$$= \cdots + z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} \boxed{b_0^2} + z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} \sum_{m=1}^{+\infty} b_m b_{-m} z_1^{-m} z_2^{m}$$

$$= \left[:\psi_p(z_1)\psi_p(z_2): - \frac{1}{2}\frac{1}{\sqrt{z_1 z_2}}\right] + \boxed{\frac{1}{2}} z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} + z_1^{-\frac{1}{2}} z_2^{-\frac{1}{2}} \sum_{m=1}^{+\infty} 1\cdot\left(\frac{z_2}{z_1}\right)^{m}$$

$$= \left[:\psi_p(z_1)\psi_p(z_2): - \frac{1}{2}\frac{1}{\sqrt{z_1 z_2}}\right] + \underbrace{\frac{1}{2}\frac{1}{\sqrt{z_1 z_2}} + \frac{1}{z_1 - z_2}\sqrt{\frac{z_2}{z_1}}}$$

绿色有 zero VEV.

$$= \left(\psi_p(z_1)\psi_p(z_2)\right) + \frac{1}{z_1 - z_2} + \underbrace{\frac{z_1 - z_2}{8 z_2^2} - \frac{(z_1 - z_2)^2}{8 z_2^3} + \cdots}_{z_1 \to z_2 \text{ 时为}0}$$

- vacuum: $b_{n>0}|0\rangle = 0$, $\langle 0| b_{n<0} = 0$, $\underbrace{n\in\mathbb{Z},}_{\text{Ramond}}$ or, $\underbrace{n\in\mathbb{Z}+\frac{1}{2}}_{\text{NS}}$

- On $\mathbb{C}$: 用 unambiguous 的 contour normal ordering 来做

$T_p(z)$ 的定义

$$T_p(w) \equiv -\frac{1}{2}\oint_w \frac{dz}{2\pi i}\frac{1}{z-w}\psi_p(z)\,\partial\psi_p(w)$$

NS: $$T_p(z) = -\frac{1}{2}\oint_w \frac{dz}{2\pi i}\frac{1}{z-w}\psi(z)\,\partial\psi(w)$$

$$= -\frac{1}{2}\int_w \frac{dz}{2\pi i}\frac{1}{z-w}\left(:\psi_p(z)\,\partial\psi_p(w): + \partial_w\frac{1}{z-w}\right)$$

$$= \underbrace{-\frac{1}{2}:\psi_p\,\partial\psi_p:(w)}_{\langle 0|T_p(w)|0\rangle = 0}$$

R: $$T_p(z) = -\frac{1}{2}\oint\frac{dz}{2\pi i}\frac{1}{z-w}\Big\{$$

$$\left[:\psi_p(z)\,\partial\psi_p(w): - \frac{1}{2}\partial_w\frac{1}{\sqrt{zw}}\right] + \partial_w\left[\frac{1}{2}\frac{1}{\sqrt{zw}} + \frac{1}{z-w}\sqrt{\frac{z_2}{z_1}}\right]\Big\}$$

$$= \underbrace{-\frac{1}{2}:\psi_p\,\partial\psi_p(w): - \frac{1}{8}\frac{1}{w^2}}_{\langle 0|\cdots|0\rangle = 0} + \underbrace{\frac{1}{16}\frac{1}{w^2}}_{\langle T_p(w)\rangle}$$

注: R-sector 中 $:\psi_p\partial\psi_p(w): = \cdots + \left(-\frac{1}{2}\right) b_0 b_0\, w^{-2}$ 不杀真空

- NS sector: $T(w) = -\frac{1}{2} :\psi_p(w)\, \partial\psi_p(w):$

$$R\,T(z)\,T(w) = \frac{1}{4(z-w)^4} - \frac{1}{(z-w)^2} :\psi(w)\,\partial\psi(w): - \frac{1}{z-w} :\psi(w)\,\partial^2\psi(w):$$

$$+ O(z-w)$$

$$= \frac{1}{4(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

困到 $:A(z): \quad :B(w): =$ all cross contractions (注意负号)

$$:\psi(w)\,\psi(w): \;=\; :\psi^{(n)}(w)\,\psi^{(n)}(w): \;= 0$$

形式算符代数运算 (later formalized by "VOA")

- 设算符有 mode expansion $\mathcal{O}(z) = \sum_{n\in\mathbb{Z}-h} \mathcal{O}_n z^{-n-h}$

设 ∃ "vacuum" $|0\rangle$ s.t.

$$\mathcal{O}_{n>-h}|0\rangle = 0\,, \quad \underbrace{|\mathcal{O}\rangle \equiv \mathcal{O}(0)e^{ipz}|0\rangle}_{} = \mathcal{O}_{-h}|0\rangle$$

因为当 $n=-h\in\mathbb{Z}-h$ 时

$$\mathcal{O}_{-h}z^{-(-h)-h} = \mathcal{O}_{-h}$$

再设 $\forall\, \mathcal{O}(z)$ 都与 $|\mathcal{O}\rangle$ 一一对应.

local

数学家的记法，$a\in V$ = Hilbert space

$$Y(a,z) = \sum_n a_n z^{-n-1}\,, \qquad a_{n>-1}\mathbb{1} = 0 \quad Y(a,z\to 0)\mathbb{1} = a$$

$$Y(a,z) \sim a(z) \qquad \mathbb{1} \sim |0\rangle$$

是 formal series，没有和函数.

- $$\partial\mathcal{O}(z) = \sum_n \underbrace{\mathcal{O}_n(-n-h)}_{(\partial\mathcal{O})_n} z^{-n-h-1}$$

$$\| $$

$$\sum_n (\partial\mathcal{O})_n \, z^{-n-(h+1)}$$

- $$\mathcal{O}_n = \oint_0 \frac{dz}{2\pi i} z^{h+n-1} \mathcal{O}(z)$$

$\forall$ operator $\mathcal{O}$, $\forall$包含原点的路径.

- $\mathcal{O}(z)$ $|\mathcal{O}\rangle$

$\mathcal{O}_n$

- Composite ops from $\{\cdots\}$, $(\cdots)$

- 定义 $\{\mathcal{O}_1\mathcal{O}_2\}_n(w)$ by (假设是 Laurent 级数)

$$\mathcal{O}_1(z)\,\mathcal{O}_2(w) \equiv \sum_{n\in\mathbb{Z}} \frac{\{\mathcal{O}_1\mathcal{O}_2\}_n(w)}{(z-w)^n},$$

or, $$\{\mathcal{O}_1\,\mathcal{O}_2\}_n(w) = \oint_w \frac{dz}{2\pi i}(z-w)^{n-1}\,\mathcal{O}_1(z)\,\mathcal{O}_2(w)$$

- 由于上述 OPE 是 z-w 的整幂级数，∃ $(z-w)^0$ 项

$$\mathcal{O}_1(z)\,\mathcal{O}_2(w) = \overbrace{\cdots}^{z-w\text{ 负幂}} + \underline{(\mathcal{O}_1\,\mathcal{O}_2)(w)} + \overbrace{\cdots}^{z-w\text{ 正幂}}$$

- 定义 Normal ordered propuct $(\,\cdot\,\cdot\,)$ by contour integral

$$(\mathcal{O}_1\mathcal{O}_2)(w) \equiv \{\mathcal{O}_1\mathcal{O}_2\}_0(w) \equiv \oint_w \frac{dz}{2\pi i}\frac{1}{z-w}\mathcal{O}_1(z)\mathcal{O}_2(w)$$

- $$\partial\mathcal{O}_1(z)\,\mathcal{O}_2(w) = \partial_z \sum_n \frac{\{\mathcal{O}_1\mathcal{O}_2\}_n(w)}{(z-w)^n}$$

$$= \sum_n (-n)\frac{\{\mathcal{O}_1\mathcal{O}_2\}_n(w)}{(z-w)^{n+1}}$$

$$\Rightarrow \quad (\partial\mathcal{O}_1\mathcal{O}_2)(w) = -(-1)\,\{\mathcal{O}_1\mathcal{O}_2\}_{-1}(w)$$

$$\Rightarrow \quad (\partial^n\mathcal{O}_1\mathcal{O}_2)(w) = n!\,\{\mathcal{O}_1\mathcal{O}_2\}_{-n} \qquad n\geqslant 0$$

$$(\mathcal{O}_1\mathcal{O}_2) = \{\mathcal{O}_1\mathcal{O}_2\}_0$$

$\{\ \}_{-n}$ 完全由 Normal order product $(\ \ )$ 控制.

$$\Rightarrow \quad \mathcal{O}_1(z)\,\mathcal{O}_2(w) = (z-w)\text{ 负幂}$$

$$+ \sum_{n=0}^{+\infty}\frac{1}{n!}(\partial^n\mathcal{O}_1\mathcal{O}_2)(w)\,(z-w)^n$$

• Bosonic $\mathcal{O}_i$, $(\mathcal{O}_1\mathcal{O}_2) \neq (\mathcal{O}_2\mathcal{O}_1)$ 即不满足交换律

$$(\mathcal{O}_1\mathcal{O}_2)(w) = \oint_w \frac{dz}{2\pi i} \frac{\mathcal{O}_1(z)\mathcal{O}_2(w)}{z-w}$$

$$(\mathcal{O}_2\mathcal{O}_1)(w) = \oint_w \frac{dz}{2\pi i} \frac{\mathcal{O}_2(z)\mathcal{O}_1(w)}{z-w} = \oint_w \frac{dz}{2\pi i} \frac{\mathcal{O}_1(w)\mathcal{O}_2(z)}{z-w}$$

$$= \oint_w \frac{dz}{2\pi i} \frac{1}{z-w} \sum_n \frac{\{\mathcal{O}_1\mathcal{O}_2\}_n(z)}{(w-z)^n}$$

$$= \sum_n \oint_w \frac{dz}{2\pi i} \frac{(-1)^n}{(z-w)^{n+1}} \{\mathcal{O}_1\mathcal{O}_2\}_n(z)$$

$$\Big\downarrow \text{高}$$

$$= \sum_{n\geq 0} \frac{(-1)^n}{n!} \partial^n \{\mathcal{O}_1\mathcal{O}_2\}_n(w)$$

$$\oint_w \frac{dz}{2\pi i} \frac{f(z)}{(z-w)^{n+1}} = \frac{1}{n!} f^{(n)}(w)$$

$$\Rightarrow (\mathcal{O}_1\mathcal{O}_2)(w) - (\mathcal{O}_2\mathcal{O}_1)(w) = -\sum_{n\geq 1} \frac{(-1)^n}{n!} \partial^n \{\mathcal{O}_1\mathcal{O}_2\}_n(w)$$

$$\qquad \parallel$$

$$[\mathcal{O}_1, \mathcal{O}_2](w)$$

$\Rightarrow (\mathcal{O}_1\mathcal{O}_2) = (\mathcal{O}_2\mathcal{O}_1)$

if $\mathcal{O}_1(z)\mathcal{O}_2(w)$ 没有负幂次项,

if $\mathcal{O}_1(z)\mathcal{O}_2(w)$ 只有一 pole 且 $\propto \mathbb{1}$.

• $(a(bc)) \neq ((ab)c)$ 不满足结合律

$$(ab)(w) = \oint_w \frac{dz}{2\pi i} R\, \frac{a(z)b(w)}{z-w}$$

$$= \oint_{|z|=|w|+\epsilon} \frac{dz}{2\pi i} \frac{a(z)b(w)}{z-w} - \oint_{|z|=|w|-\epsilon} \frac{dz}{2\pi i} \frac{b(w)\;a(z)}{z-w}$$

$$= \oint_{|z|=|w|+\epsilon} \frac{dz}{2\pi i} \frac{1}{z-w} \sum_{m,n} a_m z^{-m-h_a} b_n w^{-n-h_b}$$

$$- \oint_{|z|=|w|-\epsilon} \frac{dz}{2\pi i} \frac{1}{z-w} \sum_{m,n} b_n w^{-n-h_b} a_m z^{-m-h_a}$$

① $(z-w)^{-1}$ 根据 $|z| \gtrless |w|$ 展开为 $\frac{z}{w}$ 或 $\frac{w}{z}$ 的级数

② $\oint \frac{dz}{2\pi i} \frac{1}{z^n} = \delta_{n,1}$

$$= \sum_N \underbrace{\left(\sum_{m \leq -h_a} a_m b_{N-m} + \sum_{m \geq -h_a+1} b_{N-m} a_m \right)}_{(ab)_N} w^{-N-h_a-h_b}$$

$$\Rightarrow \quad (ab)_n = \sum_{m \leq -h_a} a_m b_{n-m} + \sum_{m \geq -h_A+1} b_{n-m} a_m$$

$$(ab)_{-h_a-h_b} = a_{-h_a} b_{-h_b} + a_{-h_a-1} b_{-h_b+1} + \cdots$$
$$+ b_{-h_b-1} a_{-h_a+1} + \cdots$$

$$(ab)(0)|0\rangle = (ab)_{-h_a-h_b}|0\rangle = a_{-h_a} b_{-h_b}|0\rangle$$

$$\Rightarrow \quad (a(bc))(0)|0\rangle = a_{-h_a} b_{-h_b} c_{-h_c}|0\rangle$$

$$\#$$

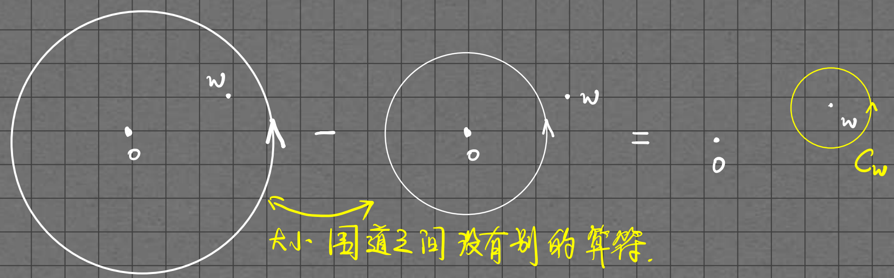
$$((ab)c)(0)|0\rangle = a_{-h_a} b_{-h_b} c_{-h_c}|0\rangle + \underbrace{\cdots}$$

没有结合律.

- 考虑 bosonic a, b , $A \equiv \oint_0 \frac{dz}{2\pi i} a(z)$, 利用积分路径不确定

$$[A, b(w)] \equiv A\, b(w) - b(w)\, A$$

$$\equiv \oint_{|w|+\epsilon} \frac{dz}{2\pi i} a(z)\, b(w) - \oint_{|w|-\epsilon} \frac{dz}{2\pi i} b(w)\, a(z) = \oint_w \frac{dz}{2\pi i} R\, a(z) b(w)$$



$$= \oint_w R\ a(z)\, b(w)\, dz .$$

w附近的 contour.

- $B \equiv \oint_0 \frac{dz}{2\pi i} b(w) \Rightarrow [A, B] = \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} R\, a(z) b(w)$

- 设 $R\, a(z) b(w) = \cdots + \frac{\{ab\}_1(w)}{z-w} + \cdots = R\, b(w) a(z)$

则

$$[A, B] = \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} \left(\cdots + \frac{\{ab\}_1(w)}{z-w} + \cdots \right)$$

$$= \oint_0 \frac{dw}{2\pi i} \{ab\}_1(w) = \underset{w\to 0}{\mathrm{Res}} \{ab\}_1(w)$$

$$[B, A] = \oint_0 \frac{dz}{2\pi i} \oint_z \frac{dw}{2\pi i} R\, b(w) a(z)$$

$$= \oint_0 \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{\{ab\}_1(w)}{z-w}$$

$$= \oint_0 \frac{dz}{2\pi i} \left(- \{ab\}_1(z) \right) = - \underset{z\to 0}{\mathrm{Res}} \{ab\}_1(z)$$

$$\Longrightarrow [A, B] = - [B, A]$$

• 例: Primary $\mathcal{O}(z) = \sum_n \mathcal{O}_n z^{-n-h}$, $T(z) = \sum_n L_n z^{-n-2}$

$$[L_n, \mathcal{O}(w)] = \oint_w \frac{dz}{2\pi i} z^{n+1} T(z) \mathcal{O}(w)$$

$$= [w^{n+1}\partial_w + (n+1) h w^n] \mathcal{O}(w)$$

$$\Rightarrow \quad [L_0, \mathcal{O}(w)] = (w\partial_w + h) \mathcal{O}(w)$$

$$[L_{-1}, \mathcal{O}(w)] = \partial_w \mathcal{O}(w) + 0$$

$$[L_1, \mathcal{O}(w)] = (w^2\partial_w + 2hw) \mathcal{O}(w)$$

打到 $|0\rangle$ 上

$$\Rightarrow \quad L_0|\mathcal{O}\rangle = h|\mathcal{O}\rangle, \quad \underline{L_{-1}}|\mathcal{O}\rangle = |\partial\mathcal{O}\rangle,$$

$$L_{n\geq 1}|\mathcal{O}\rangle = 0$$

for $|\mathcal{O}\rangle \equiv \mathcal{O}(0)|0\rangle$

L_0 ~ Dilatation $\qquad$ L_{-1} ~ translation

L_{+1} ~ special CF

$\bar{L}_0$ $\qquad$ $\bar{L}_{-1}$

$\bar{L}_{+1}$

• 例： $T(z) = \sum_n L_n z^{-n-2}$, $L_m = \oint \frac{dz}{2\pi i} z^{m+1} T(z)$

$$[L_m, L_n] = \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{m+1} w^{n+1} T(z) T(w)$$

$$= \oint_0 \frac{dw}{2\pi i} w^{n+1} \oint_w \frac{dz}{2\pi i} z^{m+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \right]$$

利用 高阶导数公式 $\left(\partial_z z^{m+1}\right)''' \big|_w \to (m+1) m (m-1) w^{m+1-3}$

$$= \underbrace{(m-n) L_{m+n} + \frac{c}{12} (m-1) m (m+1) \delta_{m+n,0}}$$

Virasoro 代数 in terms of L_m

• Primary $\mathcal{O}(z) = \sum_n \mathcal{O}_n z^{-n-h}$

$$[L_m, \mathcal{O}_n] = \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{m+1} w^{n+h-1} T(z) \mathcal{O}(w)$$

$$= \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{m+1} w^{n+h-1} \left[\frac{2\mathcal{O}(w)}{(z-w)^2} + \frac{\partial \mathcal{O}(w)}{z-w} \right]$$

$$= (mh - m - n) \phi_{m+n}$$

$$\Rightarrow [L_0, \mathcal{O}_n] = \underbrace{-n}_{\text{weight of } \mathcal{O}_n} \mathcal{O}_n$$

- Shifted modes $\mathcal{O}(z) = \sum_n \mathcal{O}_n(w)(z-w)^{-n-h}$

$$\Rightarrow \underbrace{\mathcal{O}_n(w)}_{\text{new operator}} = \oint_w \frac{dz}{2\pi i}(z-w)^{n+h-1}\mathcal{O}(z)$$

- 定义 $(\mathcal{O}_n\mathcal{O}')(w) \equiv \{\mathcal{O}\mathcal{O}'\}_{n+h}(w)$

$$= \oint_w \frac{dz}{2\pi i}(z-w)^{n+h-1}\mathcal{O}(z)\mathcal{O}'(w)$$

$$\Rightarrow (\mathcal{O}_{-n-h}\mathcal{O}')(w) = \frac{1}{n!}(\partial^n\mathcal{O}\,\mathcal{O}')(w) \qquad n\geq 0$$

- 当 $\mathcal{O}=T$，$h=2$， $(L_{-n}\mathcal{O})(w)$ 称为 $\mathcal{O}$ 的 Virasoro descendant.

$$\Rightarrow (L_{-n-2}\mathcal{O})(w) = \frac{1}{n!}(\partial^n T\,\mathcal{O})(w)$$

$$\Rightarrow (L_{-n-2}\mathbb{I})(w) = \frac{1}{n!}\partial^n T(w)$$

$(L_{-2}\mathbb{I})(w) = T(w)$， $T(w)$ 是 $\mathbb{I}$ 的 Virasoro descendant.

- 给定 $a(z)$, $b(w)$, $a(z) = \sum_n a_n z^{-n-h_a}$

  $|b\rangle = b(0)|0\rangle$

- 定义 复合场 $(a_n b)(w) \equiv \{ab\}_{n+h_a}(w)$

  $\Rightarrow (a_n b)(w \to 0)|0\rangle = a_n |b\rangle$

- $(a_n b)(w) = \oint_w \frac{dz}{2\pi i} (z-w)^{n+h_a-1} a(z) b(w)$

  $\Rightarrow (a_{-n-h} b)(w) = \frac{1}{n!} (\partial^n a \, b)(w) \qquad n \geq 0$

$(b\partial c)(z)\,(bc)(w)$

$= \frac{1}{z-w} \partial_z \frac{1}{z-w} + \frac{\partial c b}{z-w} +$

$+ \partial_z \frac{1}{z-w} (bc)(w)$

- 当 $a = T$, $a_n = L_n$

- $(L_{n\leq -1}\, b)(w)$ 称为 $b(w)$ 的 Virasoro - descendant.

  $(L_{-1} b)(w)$ 称为 $b(w)$ 的 $SL(2,\mathbb{C})$ descendant.

  (Note: $(L_{-1} b)(0)|0\rangle = L_{-1}\, b(0)|0\rangle = [L_{-1}, b(0)]|0\rangle$
  $= \partial b(0)|0\rangle$)

- $(L_{-2}\,\mathbb{I})(w) = \frac{1}{0!} (\partial^0 T \,\mathbb{I})(w) = T(w)$

  $T(w)$ 是 $\mathbb{I}$ 的 Virasoro descendant.