Virasoro Verma module

- Virasoro algebra 是最简单的一类 VOA

- Virasoro algebra Vir_c 由 $T(z)$ (and $\mathbb{1}$) 生成.

$$T(z)=\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$[L_m, L_n]=(m-n)L_{m+n}+\frac{c}{12}m(m-1)(m+1)\delta_{m+n,0}$$

c = central charge of Vir_c

- 假设存在 "unique vacum" $|h=0\rangle \equiv |0\rangle \equiv \Omega \equiv \mathbb{1}$, s.t.

$$L_{n\geqslant -1}|0\rangle = 0.$$

$\Rightarrow$ $SL(2,\mathbb{C})$ invariant $L_{\pm,0}|0\rangle=0$

- 定义：primary state $|h\rangle$ with conformal weight

$$L_0|h\rangle = h|h\rangle \qquad L_{n>0}|h\rangle = 0$$

- 定义：从 primary state $|h\rangle$ 出发，记

$$V_{c,h} = \text{span}\{\text{all } L_{-n_1}L_{-n_2}\cdots|h\rangle,\ 0<n_1\leq n_2\leq\cdots\}$$

称为 $|h\rangle$ 对应的 Verma module of Vir. alg. (representation)

Note:
$$L_{-n_1}L_{-n_2}\chi = [L_{-n_1}, L_{-n_2}]\chi + L_{-n_2}L_{-n_1}\chi$$
$$= [(-n_1)-(-n_2)]L_{-n_1-n_2}\chi + L_{-n_2}L_{-n_1}\chi$$

- $L_{-n_1}L_{-n_2}\cdots|h\rangle$ 称为 $|h\rangle$ 的 Virasoro descendant states

定义 level $N=\sum_i n_i$

-
$$L_0\left(L_{-n_1}L_{-n_2}\cdots|h\rangle\right) = \left(h+\sum_i n_i\right)L_{-n_1}L_{-n_2}\cdots|h\rangle$$
$$= (h+N)L_{-n_1}\cdots|h\rangle$$

	level	L_0-eigenvalue	#
$\vert h\rangle$	0	h	1
$L_{-1}\vert h\rangle$	1	$h+1$	1
$L_{-1}^2\vert h\rangle$, $L_{-2}\vert h\rangle$	2	$h+2$	2
$L_{-1}^3\vert h\rangle$ $L_{-1}L_{-2}\vert h\rangle$, $L_{-3}\vert h\rangle$	3	$h+3$	3
$L_{-1}^4\vert h\rangle$, $L_{-1}^2L_{-2}\vert h\rangle$, $L_{-1}L_{-3}\vert h\rangle$ $L_{-2}^2\vert h\rangle$ $L_{-4}\vert h\rangle$	4		5
1^5, $1^3 2$, $1^2 3$, $1 2^2$, $1 4$ $2 3$ 5	5		7
			⋮

Partition 配分 $P(N)$

- 定义 inner product $\langle\psi|\chi\rangle$ s.t. $L_n^\dagger = L_{-n}$

- Unitarity all $\langle\psi|\psi\rangle \geq 0$ $\Rightarrow$ $\underbrace{h \geq 0 \quad c \geq 0}_{\text{unitary 表示的必要条件}}$

Character.

- 定义 特征标 of Virasoro module $V_{c,h}$

$$\chi_{V_{c,h}}(q) \equiv \mathrm{tr}_{V_{c,h}}\, q^{L_0-\frac{c}{24}}$$

(无穷维线性空间求迹)

$q \to$ counting param / fugacity.

$$= \sum_{N=0} \underbrace{(\text{dim Level } N \text{ space})}\, q^{h+N-\frac{c}{24}}$$

$$= q^{h-\frac{c}{24}} \sum_{N=0} P(N)\, q^N$$

Note that

$$\sum_{N=0} P(N)\, q^N$$

$$= \underbrace{\frac{1}{\prod_{n=1}^{+\infty}(1-q^n)}}$$

$$= \frac{1}{(q;q)} = \frac{q^{\frac{1}{24}}}{\underbrace{\eta(\tau)}}, \quad q = e^{2\pi i\tau}$$

q-Pochhammer symbol

Dedekind η-function

$$(z;q) \equiv \prod_{k=0}^{+\infty}(1-zq^k)$$

$$\eta\left(-\frac{1}{\tau}\right) = \tau^{\frac{1}{2}}\, \eta(\tau)$$

$\mathrm{Li}(z)$.

Null states, singular vector

- 定义 singular state / null state 为 $V_{c,h}$ 中既 primary 又 descendant 的 level N 态

即 $$\chi = \sum_{n_1+\cdots=N} c(\{n\}) L_{-n_1}\cdots|h\rangle$$

$$L_{n\geq 1}\,\chi = 0$$

① $V_{c,h}$ 是 Vir 的可约表示（χ 自己生成一个 $V_{c,h+N}$）

② null $\perp$ $V_{c,h}$ $\Rightarrow$ submodule from null $\perp$ $V_{c,h}$

e.g. $$\langle\chi|L_{-k_1}\cdots|h\rangle = \langle h|\cdots\underbrace{L_{k_1}|\chi\rangle}_{=0}{}^{\dagger}=0$$

$$\Downarrow$$

$$\langle\chi|\chi\rangle = 0$$

③ Existence encoded in level N Kac-determinant

$$\underbrace{\det M^{(N)}_{ij}}_{\text{Kac det}} \equiv \det_{ij}\langle i|j\rangle$$

where $|i\rangle, |j\rangle$ span level N subspace.

$\det M^{(N)} = 0 \Rightarrow$ 存在 null 或 null 的 descendant

· $M^{(N)}$ depends on N, h, c (Kac)

$$\det M^{(N)} \propto \prod_{\substack{r,s=1 \\ rs\le N}} \left(h - h_{r,s}(c)\right)^{P(N-rs)}$$

$$h_{r,s}(c) \equiv \frac{c-1}{24} + \frac{1}{4}\left(r\alpha_+ + s\alpha_-\right)^2$$

$$\alpha_\pm \equiv \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}.$$

① At generic h, c, $h \neq h_{r,s}(c)$, $\det M^{(N)} \neq 0$, $\forall N$.

此时 $V_{c,h}$ 中 没有 null state.

② At generic c, if $h = h_{r,s}(c)$ for some $r, s \in \mathbb{N}$

则 $\underbrace{\det M^{(N<rs)}} \neq 0$ $\quad \det M^{(N=rs)} = 0$

$$= \prod_{\substack{r',s'=1 \\ r's'\le N<rs}} \left(h_{r,s}(c) - h_{r',s'}(c)\right)^{P(N-rs)}$$

$\Rightarrow$ level $N<rs$ 时，没有 null

null χ appears at level $N=rs$

③ At generic c. $h = h_{r,s}(c)$

$$h[\chi] = h_{r,s}(c) + rs \neq h_{r',s'}(c)$$

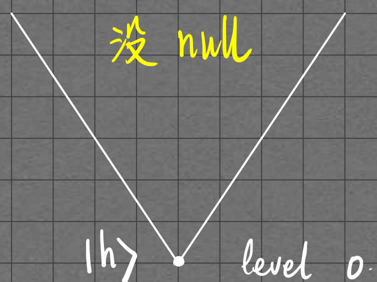
$\Rightarrow$ $\chi^{(N)}$ 生成的 Verma submodule 没有 null 了.

④ $N = rs$ 往下每个 level 都有 χ 的 Virasoro descendants

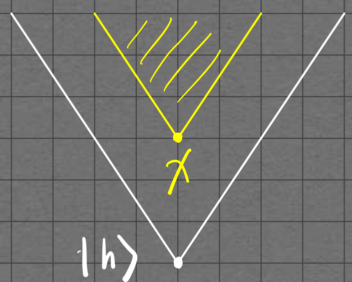
At generic c, $h = h_{r,s}(c)$, level $N' > rs$ 有 $P(N'-rs)$ 个 χ 的 descendants ; 它们对应

$$\det M^{(N')} = \prod_{\substack{r',s'=1 \\ r's' \leq N'}} (h - h_{rs}(c))^{P(N'-r's')}$$

的 $P(N'-rs)$-阶零点、$h = h_{rs}(c)$



generic h, c



$p(N-rs)$ 个 χ-descendants.

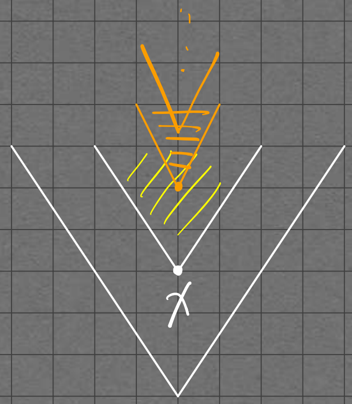
level rs

level 0.

generic c

special $h = h_{r,s}(c)$

$V_{c,h}$ reducible.



special c

special $h = h_{r,s}(c)$

$V_{c,h}$ highly reducible

null 所生成的 Verma module 里又有 null

$$c(p,p') = 1 - \frac{6(p-p')^2}{p\,p'}$$

$p > p'$ 互质

"minimal model $M(p,p')$"

(Lee-Yang, crit. Ising, ⋯)

- Irreducible module $M_{c,h} = V_{c,h} /$ nulls. Verma modules.

- $\langle \chi(z) \prod_{j=1}^{n} \phi_{h_j}(z_j) \rangle = 0$

$\parallel$

$L_{-*} \cdots \phi_{h_{r,s}}(z)$

$\Downarrow$

some ODE of $\underbrace{\langle \phi_{h_{r,s}}(z)\, \phi_{h_1}(z_1)\, \phi_{h_2}(z_2) \rangle}$

constrained. non-vanish only when h_j and $h_{r,s}$ are related.

- $\mathrm{tr}\, \underbrace{\chi(z)}_{\text{some null}}\, q^{L_0 - \frac{c}{24}} = 0 \Rightarrow$ "modular differential equations"

Special c

- 考虑互质自然数 $p>p'$，考虑特殊 c

$$\underbrace{c = 1 - \frac{6(p-p')^2}{pp'}}_{Vir(p,p')} , \quad h_{r,s}(c) \equiv \frac{c-1}{24} + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2$$

$$= \frac{(pr-p's)^2-(p-p')^2}{4pp'}$$

① $p-p'=1$, $p' \in \mathbb{Z}_{\geq 2}$

$$1 > c = 1 - \frac{6}{(p'+1)p'} \geq 0$$

$$h_{r,s}(c) = \frac{[(p'+1)r - p's]^2 - 1}{4p'(p'+1)} > 0$$

$\Rightarrow$ unitary, reducible $V_{c,h}$

② general $p > p' \in \mathbb{Z}_{\geq 2}$, non-unitary minimal models.

$h_{min} = h_{r_0,s_0} < 0$, where $pr_0 - p's_0 = 1$

Rational CFT

上面这些 c 对应 minimal models $M_{p,p'}$, Hilbert space

$$\mathcal{M}_{p,q} = \bigoplus_{\substack{1\leq r\leq p'-1 \\ 1\leq s\leq p-1}} M_{r,s} \otimes \overline{M}_{r,s} \qquad \phi_{(r,s)(r,s)}(z,\bar{z})$$

finite number, closed under fusion / OPE.

$$h(\chi) = \boxed{h_{r,s}(c) + rs} = h_{r',s'}$$

$$\phi_{r,s} \, \phi_{r',s'} = \sum_{r'',s''} \phi_{r'',s''}$$

$$\bigoplus N_{(r,s),(r',s')} M_{r,s} \otimes \overline{M}_{r',s'}$$

$\downarrow$

$SL(2,\mathbb{Z})$ inv. partition

$$V_{c_{p,q},\, h_{r,s}(c_{p,q})} \longrightarrow M_{c_{p,q}\; h_{r,s}(c_{p,q})}$$

$\downarrow$

$$ch_{(p,q)(r,s)}(\tau) \xrightarrow{\tau \to -\frac{1}{\tau}} ch_{(p,q)(r,s)}\left(-\frac{1}{\tau}\right)$$

$\|$

$$\sum_{r',s'} S^{r',s'}_{r,s}\, ch_{p,q\,(r',s')}(\tau)$$

$\tau = i\beta$

$\downarrow \beta \to +0$

High temp.

dim. red

- $c = 1 - \frac{6(p-p')^2}{pp'}$, $h_{r,s}(c) = \frac{(pr-p's)^2-(p-p')^2}{4pp'}$

回顾 $\alpha_\pm \equiv \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}$

$$\Rightarrow \alpha_+ + \alpha_- = \frac{2\sqrt{1-c}}{\sqrt{24}}, \quad \alpha_+\alpha_- = -1$$

$$= \sqrt{\frac{6(p-p')^2}{6pp'}}$$

$$\overset{p>p'}{=} \frac{p-p'}{\sqrt{pp'}}$$

$$\alpha_+ = \sqrt{\frac{p}{p'}} \qquad \alpha_- = -\sqrt{\frac{p'}{p}}$$

- 定义 Q^2 by $c \equiv 1+6Q^2 \Rightarrow Q^2 = -\frac{(p-p')^2}{pp'}$

$$\Rightarrow Q = i(\alpha_+ + \alpha_-)$$

定义 $b \equiv i\alpha_+ = i\sqrt{\frac{p}{p'}}$, $b^{-1} \equiv i\alpha_- = -i\sqrt{\frac{p'}{p}} \Rightarrow bb^{-1} = i^2\alpha_+\alpha_- = 1$

$$\Rightarrow Q = b + b^{-1}$$

$$\Rightarrow h_{r,s} = \frac{c-1}{24} + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2$$

$$= \frac{(b+b)^2}{4} + \frac{1}{4}(-irb - isb^{-1})^2$$

$$= \frac{1}{4}(b+b)^2 - \frac{1}{4}(rb+sb^{-1})^2$$

$$= \left(\frac{b+b^{-1}+rb+sb}{2}\right)\left(\frac{b+b^{-1}-rb-sb^{-1}}{2}\right)$$

定义 $\alpha_{r,s} \equiv \frac{1}{2}(1-r)b + \frac{1}{2}(1-s)b$, 则

$$h_{r,s} = \alpha_{r,s}(Q - \alpha_{r,s})$$

Lee-Yang model

- $p=5,\ p'=2$, $c=-\frac{22}{5}$ $r=1,$ $s=1,2,3,4$.

$h_{1,1}=0$ $h_{1,2}=-\frac{1}{5}$ $h_{1,3}=-\frac{1}{5}$ $h_{1,4}=0$

对应 vacuum

- $V_{c,h_{1,1}}=V_{c,h_{1,4}}$, 有 level 1 和 4 的 null

$\chi^{(1)}=L_{-1}|h_{1,1}\rangle,$ $\chi^{(4)}=(L_{-2})^2|0\rangle-\frac{5}{3}L_{-4}|0\rangle$

满足 $L_{n\geq 0}\chi^{(i)}=0$

这个 null 商掉之后, $|h_{1,1}\rangle$ 就相当于 $|0\rangle$ 了. 因

$L_{n\geq -1}|h_{1,1}\rangle\sim 0$ after quotient.

$V_{c,h_{1,2}}=V_{c,h_{1,3}}$, 有 level 2, 3 的 null.

- Comepond to non-hermitian LG model (CP inv, real spectrum)

$$\frac{1}{2}(\partial_\mu\phi)^2+i(h-h_0)\phi+ig\phi^3$$

one relevant op : $\phi\sim\phi_{1,2}$

- The LG model describe the continuous limit of feromagnetic Ising model

Ising model

- Ising model at criticality :

  Based on a $Vir_{c(4,3)}$ with $p=4$, $p'=3$, $c(4,3)=\frac{1}{2}$

- $1 \le r \le 2$, $1 \le s \le 3$

-

h	$r=1$	$r=2$
$s=1$	0	$\frac{1}{2}$
$s=2$	$\frac{1}{16}$	$\frac{1}{16}$
$s=3$	$\frac{1}{2}$	0

3 个 Virasoro primaries

$\phi_{1,2}$ σ $\phi_{1,3}$ ε

- $\sigma \times \sigma = id + \varepsilon$ $\sigma \times \varepsilon = \sigma$ $\varepsilon \times \varepsilon = id$

- $Vir_{c(4,3)}$ algebra can be realized by real fermions ψ.

  $T = -\frac{1}{2}\psi\partial\psi$

- ψ itself is the primary with $h=\frac{1}{2}$

- Ising model :

energy $\varepsilon(z,\bar{z}) = \psi(z)\,\bar{\psi}(\bar{z})$ $(h,\bar{h}) = (\frac{1}{2},\frac{1}{2})$

spin $\sigma(z,\bar{z}) = \phi_{(1,2)}(z)\,\phi_{1,2}(\bar{z})$ $(h,\bar{h}) = (\frac{1}{16},\frac{1}{16})$

VOA

- Defining Data

① V : 线性空间 (space of states)

② unique vacuum $\mathbb{1} = |0\rangle = \Omega \in V$,

conformal element $T \in V$

③ state/op 对应: $Y: V \to \mathrm{End}\, V((z))$, s.t.

$$Y(a,z) = \sum a_n z^{-n-h_a}, \quad a_n : V \to V.$$

④ translation $L_{-1}: V \to V$, and related to T.

$$Y(T,z) = \sum_n L_n z^{-n-2}, \quad L_{-1} \text{ is the translation}$$

and L_m form the Virasoro algebra with some c_{2d}

L_0 on V is diagonalizable and bounded from below.

$L_0 \mathbb{1} = 0 \qquad L_0 T = 2$

⑤ $Y(\mathbb{1}, z) = \mathrm{id} : V \to V$, $\quad Y(a,z)\mathbb{1} = a + O(z)$ (单射)

$$[L_{-1}, Y(a,z)] = \partial Y(a,z)$$

Jacobi identities

Chiral boson

- $X(z)X(w) \sim -\log(z-w)$

  $\partial X(z)\partial X(w) \sim \frac{1}{(z-w)^2}$

- Virasoro algebra appears as one subalgebra of the algebra of boson theory

$$T(z) = -\frac{1}{2}(\partial X\partial X)(z) + \sqrt{2}i\alpha_0\partial^2 X(z) \Rightarrow c = 1-24\alpha_0^2$$

- $V_\alpha \equiv :e^{i\sqrt{2}\alpha X}:$ 是 Virasoro primary:

$$T(z)V_\alpha(w) \sim \frac{\alpha(\alpha-2\alpha_0)}{(z-w)^2}V_\alpha(w) + \frac{\partial V_\alpha(w)}{z-w}$$

- 考虑 $V_{\alpha_{r,s}}$，其中

$$\alpha_{r,s} = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_-$$

$$\Rightarrow h[V_{\alpha_{r,s}}] = \alpha_{r,s}^2 - 2\alpha_0\alpha_{r,s}$$

$$= \frac{c-1}{24} + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 = h_{r,s}$$

$\Rightarrow$ $V_{\alpha_{r,s}}$ 是对应 reducible Virasoro modules 的 primaries.

称为 degenerate vertex operators.

- Consider $\alpha_0 = 0$ $\quad T = -\frac{1}{2}(\partial X \partial X)$, $L_0 = \frac{1}{2} a_0^2 + \sum_{n>0} a_{-n} a_n$

  $c = 1$

- ∂X 及其各种求导、normal ordered product 构成一个 VOA，称为 Heisenberg VOA $\mathbb{H}$

$$\partial X = \sum_n a_n z^{-n-1}$$

- 对 $\forall\ \alpha \in \mathbb{R}$，可对 $V_\alpha|0\rangle$ 作用所有 a_{-n} 的组合由此生成 $\mathbb{H}$ 的不可约表示 M_α Heisenberg VOA

$$M_\alpha \equiv \{ a_{-n_1} \cdots V_\alpha|0\rangle \mid 1 \leq n_1 < n_2 < \cdots \}$$

- $L_0 (a_{-n_1} \cdots V_\alpha(0)|0\rangle) = \alpha^2 + (n_1 + \cdots)$

- M_0 的特征标 (character)

$$e^{2\pi i \tau} = q.$$

$$ch_{M_0}(q) = \mathrm{tr}\, q^{L_0 - \frac{c_{2d}}{24}} = q^{-\frac{1}{24}} \sum_{N=0}^{+\infty} P(N)\, q^N = \frac{1}{\eta(\tau)}$$

$P(N)$ counts $a_{-k_1} \cdots a_{-k_\ell} |0\rangle$

$k_1 + \cdots + k_\ell = N.$

- $ch_{M_0}(q)$ not modular invariant:

$$ch_{M_0}(e^{-2\pi i \frac{1}{\tau}}) \neq ch_{M_0}(e^{2\pi i \tau})$$

- 满足 "quasi-modular" eq.

$$q\partial_q\, ch_{M_0} = \frac{1}{2} \underbrace{E_2(q)}_{\text{2nd Eisenstein series}}\, ch_{M_0} \Rightarrow \left(q\partial_q - \frac{1}{2} E_2\right) ch_{M_0}(q) = 0$$

$\Updownarrow$ Zhu's recursion formula

$$\mathrm{tr}_{M_0}\Big(\underbrace{T(z) + \frac{1}{2}\partial X \partial X(z)}_{=0,\ \text{by definition}}\Big)\, q^{L_0 - \frac{c}{24}} = 0$$

- $$ch_{M_\alpha}(q) = q^{\alpha^2} \frac{1}{\eta(\tau)}$$

• ∂X 是一个 conserved current $J_0 \equiv \frac{1}{\sqrt{2}} \partial X$。,

$$J_0[\partial X] = 0 \Rightarrow J_0[a_n] = 0 \qquad J_0[V_\alpha] = \alpha$$

可定义 flavored refined character $ch_M(x,q) = tr_M \, x^{J_0} q^{L_0 - \frac{c}{24}}$

$$\Rightarrow \quad ch_{M_\alpha}(x,q) = x^\alpha q^{\alpha^2} \frac{1}{\eta(q)}$$

满足 (for all α)

$$\left(q\partial_q - \frac{1}{2}E_2 + 3x\frac{\partial}{\partial x}\right) ch_{M_\alpha}(x,q) = 0$$

$$\Big\downarrow \ \alpha = 0 \qquad \frac{\partial}{\partial x} ch_{M_0}(x,q) = 0$$

$$\left(D_q^{(1)} - \frac{1}{2}E_2\right) ch_{M_0}(q) = 0$$ (Not universal equations)

$$tr_{M_0}\left(T(z) + \frac{1}{2}\partial X \partial X(z)\right) X^{J_0} q^{L_0 - \frac{c}{24}} = 0$$

• Modular inv. partition fn.

$$\int_{-\infty}^{+\infty} d\alpha \; ch_{M_\alpha}(q)\, ch_{\overline{M_\alpha}}(\bar q)$$

$$= \int_{-\infty}^{+\infty} (q\bar q)^{\alpha^2} \frac{1}{|\eta(\tau)|^2}$$

$$= \frac{1}{|\eta(\tau)|^2} \int_{-\infty}^{+\infty} e^{2\pi i \, 2i \operatorname{Im}\tau \, \alpha^2}$$

$$\propto \frac{1}{\sqrt{\operatorname{Im}\tau}} \frac{1}{|\eta(\tau)|^2}$$ SL(2, $\mathbb{Z}$) inv.

$\beta\gamma$ system

- OPE $\quad \beta(z)\gamma(w) \sim (z-w)^{-1} + \text{reg}.$

- 由于 $\beta(z)\gamma(w)$ 只有一个 pole 且 $\propto 1$. $(\beta\gamma) = (\gamma\beta)$,

  同样地. $\partial^n\beta(z)\partial^m\gamma(w)$ 只有一个 pole $\propto 1$, $(\partial^m\gamma\,\partial^n\beta) = (\partial^n\beta\,\partial^m\gamma)$

- U(1) current $\quad J(z) \equiv (\beta\gamma) \Rightarrow \partial J(z) = (\partial\beta\,\gamma)(z) + (\beta\,\partial\gamma)(z)$

$$\Rightarrow J(z)J(w) \sim -\frac{1}{(z-w)^2} + \text{reg}$$

$$J(z)\beta(w) \sim -\frac{\beta(w)}{z-w} + \text{reg}$$

$$J(z)\gamma(w) \sim \frac{\gamma(w)}{z-w} + \text{reg}.$$

- $T(z) = (1-\lambda)(\beta\partial\gamma)(z) - \lambda(\partial\beta\,\gamma)(z)$

$$= (\beta\partial\gamma)(z) - \lambda[(\beta\partial\gamma) + (\partial\beta\,\gamma)]$$

$$= T_0(z) - \lambda\,\partial J(z)$$ Virasoro subalgebras.

- $C_\lambda = 2(6\lambda(\lambda-1)+1)$; $C_0 = 2$, $C_{\lambda=\frac{1}{2}} = -1$

- $h[\beta] = 1-\lambda \qquad h[\gamma] = \lambda$

  $J_0[\beta] = -1 \qquad J_0[\gamma] = +1$

vacuum

$$\Rightarrow ch_0 = q^{-\frac{c}{24}} \underbrace{PE\left[\frac{a q^{1-\lambda}}{1-q} + \frac{a^{-1} q^{\lambda}}{1-q}\right]}_{} = q^{-\frac{c}{24}} \frac{1}{(aq^{1-\lambda};q)(a^{-1}q^{\lambda};q)}$$

Count all words of the form $\beta_{-n_1} \cdots \gamma_{-m_i} \cdots |0\rangle$

single letter character of β $\quad \frac{a q^{1-\lambda}}{1-q}$ counts $\beta_{-h_\beta}, \beta_{-h_\beta-1}, \cdots$
$\quad a q^{1-\lambda} \quad a q^{1-\lambda+1}, \cdots$

$\gamma \quad \frac{a^{-1} q^{\lambda}}{1-q}$ counts $c_{-\lambda}, c_{-\lambda-1}, \cdots$
$\quad a^{-1} q^{\lambda} \quad a^{-1} q^{\lambda+1}, \cdots$

symplectic boson

① $ch_0(\lambda = \frac{1}{2}) = q^{\frac{1}{24}} \frac{1}{(aq^{\frac{1}{2}};q)(a^{-1}q^{\frac{1}{2}};q)} = \left(\frac{\eta(\tau)}{\vartheta_4(\hat{a}|\tau)}\right)$ $\quad q\partial\partial_q$

② $ch_0(\lambda = 0) = q^{-\frac{1}{12}} \frac{1}{(aq;q)(a^{-1};q)} = q^{-\frac{1}{12}} \frac{-i a^{\frac{1}{2}} q^{\frac{1}{8}} (q;q)}{-i a^{\frac{1}{2}} q^{\frac{1}{8}} (q;q)(aq;q)(a^{-1};q)}$

$$= \frac{-i a^{\frac{1}{2}} \eta(\tau)}{\vartheta_1(\hat{a}|\tau)}$$

bc ghost

- b, c, fermionic
- $b(z)\, c(w) \sim \frac{1}{z-w} \qquad c(z)\, b(w) \sim \frac{1}{z-w}$
- $J \equiv -(bc) = (cb) = \sum_n J_n z^{-n-1}$
- $T = (\lambda-1)(b\partial c) - \lambda(c\partial b)$

$$= -b\partial c + \lambda \partial(bc) = \sum_n L_n z^{-n-2}$$

$$\Rightarrow \quad h_b = 1-\lambda \qquad h_c = \lambda, \qquad c = -12\lambda(\lambda-1) - 2$$

① when $\lambda = 0$ $\quad h_b = 1 \quad h_c = 0, \quad c = -2$

② when $\lambda = -1$ $\quad h_b = 2 \quad h_c = -1, \quad c = -26$

- $b(z) = \sum_{m\in\mathbb{Z}} b_m z^{-m-h_b} \qquad c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n-h_\beta}$

$$\Rightarrow \quad b_m = \oint_0 \frac{dz}{2\pi i} z^{h_\beta + m - 1} b(z)$$

$$c_m = \oint_0 \frac{dz}{2\pi i} z^{h_c + m - 1} c(z)$$

$$\Rightarrow \{b_m, c_n\} = \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} z^{h_\beta + m - 1} w^{h_c + n - 1} \overbrace{b(z)\, c(w)}^{\frac{1}{z-w}}$$

$$= \oint_0 \frac{dw}{2\pi i} w^{+m+n-1} = \delta_{m+n,0}$$

- vacuum module : $b_{n\leq -h_b}$ $c_{n\leq -h_c}$ acting on $|0\rangle = \mathbb{1}$

$\updownarrow$

all normal ordered products of $\partial^m b$ $\partial^n c$

- vacuum character

$$\operatorname{str} q^{L_0 - \frac{c_{2d}}{24}} z^{J_0}$$

$$= q^{-\frac{1}{24}c_{2d}} \, PE\left[+ \frac{-q^{1-\lambda} z^{-1}}{1-q} + \frac{-q^{\lambda} z}{1-q} \right]$$

$$= q^{\frac{1}{12}} \, q^{\frac{1}{2}\lambda(\lambda-1)} (z^{-1} q^{1-\lambda}; q)(q^{\lambda} z; q)$$

$$= q^{\frac{1}{12}} q^{\frac{1}{2}\lambda(\lambda-1)} \prod_{n=0}^{+\infty} (1 - z^{-1} q^{1-\lambda} q^n)(1 - z q^{\lambda} q^n)$$

Lie Algebra

$g(X^a, X^b) = g^{ab}$

- $[J^a, J^b] = i f^{ab}{}_c J^c$ 结构常数
- $K^{ab} \equiv K(J^a, J^b)$ 是 Killing form 的分量
- θ = highest root , ρ = Weyl vector
- coweight $\lambda^\vee = \dfrac{2\lambda}{(\lambda, \lambda)}$
- $h^\vee \equiv (\theta^\vee, \rho) + 1$ = dual Coxeter #

  $h \equiv (\theta, \rho) + 1$ = Coxeter #

G	$h^\vee$
$A_{N-1} = SU(N)$	N
$B_N = SO(2N+1)$	$2N-1$
$C_N = Sp(2N, \mathbb{C})$	$N+1$
$D_N = SO(2N)$	$2N-2$
E_6	12
E_7	18
E_8	30