Lie Algebra

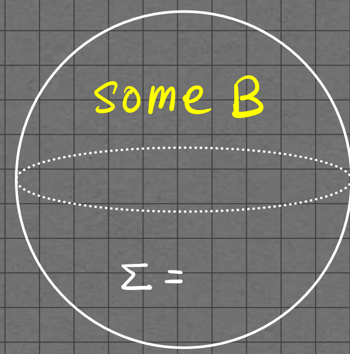
$g(X^a, X^b) = g^{ab}$

- $[J^a, J^b] = i f^{ab}{}_c J^c$ 结构常数
- $K^{ab} \equiv K(J^a, J^b)$ 是 Killing form 的分量
- θ = highest root , ρ = Weyl vector
- coweight $\lambda^\vee = \dfrac{2\lambda}{(\lambda, \lambda)}$
- $h^\vee \equiv (\theta^\vee, \rho) + 1$ = dual Coxeter #

  $h \equiv (\theta, \rho) + 1$ = Coxeter #

G	$h^\vee$
$A_{N-1} = SU(N)$	N
$B_N = SO(2N+1)$	$2N-1$
$C_N = Sp(2N, \mathbb{C})$	$N+1$
$D_N = SO(2N)$	$2N-2$
E_6	12
E_7	18
E_8	30

WZW theory and Integrable modules



- $G = SU(N)$

- $S_\Sigma = \frac{k}{16\pi}\int d^2x\, \mathrm{Tr}\, \overbrace{\partial^\mu g^{-1}\partial_\mu g}^{\geq 0}$ 动能项

  $$- \frac{ki}{24\pi}\int_B d^3y\, \epsilon^{\mu\nu\lambda}\, \mathrm{Tr}\left(g^{-1}\partial_\mu g\; g^{-1}\partial_\nu g\; g^{-1}\partial_\lambda g\right)$$

- level $k \in \mathbb{Z}$ (因 B 不唯一)

- $\mathrm{Tr}\, J^a J^b = 2\delta^{ab}$ (与表示无关，正比于 Killing form)

- EOM : $\partial_z(\underbrace{g^{-1}\partial_{\bar z} g}_{J_{\bar z}}) = 0 \Rightarrow \partial_{\bar z}(\underbrace{\partial_z g^{-1} g}_{J_z}) = 0$

对称性 $g(z,\bar z) \to \Omega(z)\, g(z,\bar z)\, \overline{\Omega}(\bar z)$ 的守恒流

- OPE (from Ward identities of $g \to \Omega(z)\, g\, \bar{\Omega}(\bar{z})$)

$$J^a(z)\, J^b(w) \sim \frac{k K^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c(w)}{z-w}$$

- $J^a(z) = \sum_m J^a_m z^{-m-1}$

$$[J^a_m, J^b_n] = i f^{ab}{}_c J^c_{m+n} + m k K^{ab} \delta_{m+n,0}$$

- Sugawara Stress tensor

$$T(z) \equiv \frac{1}{2(k+h^\vee)} \sum_{a,b} K_{ab} \left(J^a J^b\right)(z)$$

$$\Rightarrow \quad L_N = \frac{1}{2(k+h^\vee)} \sum_{a,b} \sum_{m\in\mathbb{Z}} :J^a_m J^a_{N-m}:$$

$$L_0 = \frac{1}{2(k+h^\vee)} \sum_{a,b} K_{ab} \left(J^a_0 J^b_0 + 2 \sum_{m>0} J^a_{-m} J^b_m \right)$$

$$\Rightarrow \quad c = \frac{k \dim \mathfrak{g}}{k+h^\vee}$$

$\Rightarrow$ J^a 是 Virasoro primary, $T(z) J^a(w) \sim \dfrac{J^a(w)}{(z-w)^2} + \dfrac{\partial J^a(w)}{z-w}$

- $\hat{\mathfrak{g}}_k \equiv \mathrm{span}\{ J^a_m, k, L_0 \}$ 称为 Affine Kac-Moody algebra.

(holomorphic part of WZW)

- 定义 WZW vacuum $|0\rangle$:

$$J^a_{n\geq 0}|0\rangle = 0 \qquad \bar{J}^a_{n\geq 0}|0\rangle = 0$$

AKM vacuum $|0\rangle$: $\quad J^a_{n\geq 0}|0\rangle = 0$

- 一组 WZW primary operators $\phi_{\lambda,\mu}$ ($\mathfrak{g}$-weights $\lambda\in R$, $\mu\in\tilde{R}$)

$$J^a(z)\,\phi_{\lambda,\mu}(w,\bar{w}) = -\frac{T^a_R\,\phi_{\lambda,\mu}(w,\bar{w})}{z-w} + \cdots$$

$$\bar{J}^a(\bar{z})\,\phi_{\lambda,\mu}(w,\bar{w}) = -\frac{T^a_{\tilde{R}}\,\phi_{\lambda,\mu}(w,\bar{w})}{\bar{z}-\bar{w}} + \cdots$$

- 一组 AKM primary operator / states:

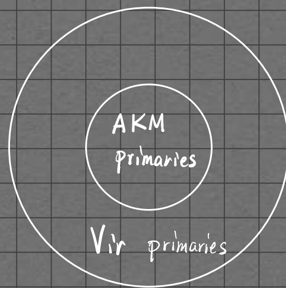
$$J^a(z)\,\phi_\lambda(w) = -\frac{T^a_R\,\phi_{\lambda,\mu}(w)}{z-w} + \cdots$$

$$|\phi_\lambda\rangle \equiv \phi_\lambda(0)|0\rangle$$

$$\Rightarrow \begin{cases} J^a_0|\phi_\lambda\rangle = -T^a_R|\phi_\lambda\rangle \\ J^a_{n>0}|\phi_\lambda\rangle = 0 \end{cases} \Rightarrow \mathrm{Span}\{|\phi_\lambda\rangle \mid \lambda\in R\} = R$$

$\mathfrak{g}$

- $$L_0 = \frac{1}{2(k+h^\vee)} \sum_{a,b} K_{ab} \left(J_0^a J_0^b + 2 \sum_{m>0} J_{-m}^a J_m^b \right)$$



$$\Rightarrow \quad L_0 |\phi_\lambda\rangle = \frac{c_2(R)}{2(k+h^\vee)} |\phi_\lambda\rangle$$

$$L_{N>0} |\phi_\lambda\rangle = 0$$

$|\phi_\lambda\rangle$ is Vir primary with

$$h = \frac{c_2(R)}{2(k+h^\vee)} = \frac{1}{2(k+h^\vee)} \sum_{a,b} K_{ab} T_R^a T_R^a$$

- 从 $\mathrm{span}\{ |\phi_\lambda\rangle \mid \lambda \in R \}$ 出发

$J^a_{m<0}$ 可产生 AKM descendants. 形成 $\hat{\mathfrak{g}}_k$ - module $\hat{R}$

$$\left.\begin{array}{ccc}
 & \mathrm{span}\{|\phi_\lambda\rangle\} = R & \\
 & \downarrow & \\
 & J_{-1}^a R & \\
 & \downarrow & \\
J_{-2}^a R & & J_{-1}^a J_{-1}^b R \\
 & \downarrow & \\
 & \vdots &
\end{array}\right\} \hat{R}$$

	$V =$ vacuum module	
$\mathbb{I}(z)$	$\vert 0\rangle$	$\text{span}\{\vert\phi_\lambda\rangle\} = \mathcal{R}$
	$\downarrow$	$\downarrow$
$J^a(z)$	$J^a_{-1}\vert 0\rangle$	$J^a_{-1}\mathcal{R}$
	$\downarrow$	$\downarrow$
$\partial J^a(z)$, $(J^aJ^a)(z)$	$J^a_{-2}\vert 0\rangle \qquad J^a_{-1}J^b_{-1}\vert 0\rangle$	$J^a_{-2}\mathcal{R} \qquad J^a_{-1}J^b_{-1}\mathcal{R}$
	$\downarrow$	$\downarrow$
	$\vdots$	$\vdots$

- $\hat{\mathfrak{g}}_k$ 中有许多 su(2) 子代数.

- $su(2)_{n,\alpha}$: $\left\{ J_3 \equiv \frac{n}{|\alpha|^2}k + \frac{\alpha\cdot H_0}{|\alpha|^2} \,,\; J_\pm \equiv E^{\pm\alpha}_{\pm n} \right\}$

  α root of $\mathfrak{g}$

- 定理.

  若相对 $\forall su(2)_{n,\alpha}$ $\hat{R}$ 总可分解为 su(2) 有限维表示直和. 则

$$(\lambda_{R}, \theta) \leq k$$

- 定义: 这样的 $\hat{R}$ 称为 $\hat{\mathfrak{g}}_k$ 的 integrable 表示

由于 $(\lambda_R, \theta) \geq 0$ 必有 $k \geq 0$.

来自 $\langle\phi_\lambda|\underbrace{E^{-\theta}_{+1}E^{+\theta}_{-1}}_{\text{Hermitian conj.}}|\phi_\lambda\rangle=\langle\phi_\lambda|J^-J^+|\phi_\lambda\rangle\geq 0$ 条件

或 从 $su(2)_{1,-\theta}$ 表示有限维性： $J^+|\phi_{\lambda_R}\rangle=E^{-\theta}_{+1}|\phi_{\lambda_R}\rangle=0$

$\Rightarrow|\phi_{\lambda_R}\rangle$ 是 $su(2)_{1,-\theta}$ 的 HW state

而 J^3 on $|\phi_{\lambda_R}\rangle$ 本征值 $\frac{2k}{|\theta|^2}-(\lambda_R,\theta^\vee)$， $\lambda'\in R_\lambda$

特别地（取 $|\theta|^2=2$）

$$k-(\lambda_R,\theta)\geq 0 \quad (\text{因为是 } su(2)_{1,-\theta} \text{ HW State})$$

- 给定 $\mathfrak{g}$ 及 $k\in\mathbb{N}_{>0}$， integrable 表示是有限的.

Admissible level

- More general affine Kac-Moody algebra. $\hat{\mathfrak{g}}_k$, $k \in \mathbb{Q}$

  相应 module 在 su(2) 子代数下可能分解成无穷维表示.

- Admissible k:

  $k + h^\vee = \frac{p}{q}$, $p, q \in \mathbb{N}$, p, q coprime

  $$p \geq \begin{cases} h^\vee & \text{if } (\text{lacing } \#, q) = 1 \\ h & \text{if } (\text{lacing } \#, q) = r^\vee \end{cases}$$

  即 p 要够大

  lacing # = 1, 2, 3 for simple $\mathfrak{g}$

  $r^\vee = 3$  G_2

- admissible reps are similar to integrable reps defined by some inequality on the HW

$sl(2)_{-\frac{1}{2}}$

- $\underbrace{-\frac{1}{2}}_{k} + \underbrace{2}_{h^\vee} = \frac{3}{2}$, $p=3$, $q=2$, $p > h, h^\vee$

$\Rightarrow k = -\frac{1}{2}$ admissible.

- can be realized from $\beta(z)\gamma(w) \sim \frac{1}{z-w}$

$$e = -\frac{1}{2}(\gamma\gamma) \qquad h = (\gamma\beta) \qquad f = \frac{1}{2}(\beta\beta)$$

- 4 admissible modules

	$[\hat{\lambda}_0, \hat{\lambda}_1]$	h	
$\hat{L}_0$	$[-\frac{1}{2}, 0]$	0	vacuum module.
$\hat{L}_{-\frac{3}{2}}$	$[1, -\frac{3}{2}]$	$-\frac{1}{8}$	
$\hat{L}_1$	$[-\frac{3}{2}, 1]$	$\frac{1}{2}$	
$\hat{L}_{-\frac{1}{2}}$	$[0, -\frac{1}{2}]$	$-\frac{1}{8}$	

$$ch_M \equiv \frac{1}{2} \, tr_M \, y^k z^{h_0} q^{L_0 - \frac{c}{24}}$$

$$ch_{\hat{L}_0} = \frac{y^{-\frac{1}{2}}}{2}\left[\frac{\eta(\tau)}{\vartheta_4(z)} + \frac{\eta(\tau)}{\vartheta_3(z)}\right]$$

$$ch_{\hat{L}_1} = \frac{y^{-\frac{1}{2}}}{2}\left[\frac{\eta(\tau)}{\vartheta_4(z)} - \frac{\eta(\tau)}{\vartheta_3(z)}\right]$$

$$ch_{\hat{L}_{-\frac{1}{2}}} = \frac{y^{-\frac{1}{2}}}{2}\left[\frac{-i\,\eta(\tau)}{\vartheta_1(z)} + \frac{\eta(\tau)}{\vartheta_2(z)}\right]$$

$$ch_{\hat{L}_{-\frac{3}{2}}} = \frac{y^{-\frac{1}{2}}}{2}\left[\frac{-i\,\eta(\tau)}{\vartheta_1(z)} - \frac{\eta(\tau)}{\vartheta_2(z)}\right]$$

$\longrightarrow$ Modular invariant P.F.

$$\sum_i |ch_{\hat{L}_i}|^2$$

- $\widehat{su}(2)_{-\frac{1}{2}}$ 对应 free HM with $\mathbb{Z}_2$ flavor symm gauged.

$$\beta \sim Q \qquad \gamma \sim \tilde{Q}$$

(see later)

- $\widehat{su}(2)_{k=-\frac{4}{3}}$ also admissible : 3 admissible modules

$[-\frac{4}{3}; 0]$ $\quad [-\frac{2}{3}; -\frac{2}{3}]$, $\quad [0, -\frac{4}{3}]$

Critical AKM

- $\hat{\mathfrak{g}}_{k=-h^\vee}$
- $T_{sug} \propto \frac{1}{k+h^\vee} \sum_{a,b} K_{ab} J^a J^b$ cannot be defined.(没有能动张量)
- L_0, L_{-1} 还是可以定义的

$$[L_0, J^a_n] = -n J^a_n, \quad [L_{-1}, J^a(z)] = \partial J^a(z)$$

- $\forall$ finite dim HWR with highest weight λ

$\longrightarrow$ $\hat{\mathfrak{g}}_{-h^\vee}$ - module M_λ

- M_λ Characters are known (Arakawa)

$$ch_\lambda = \frac{\sum_w e^{w\cdot\lambda}}{\prod_{\hat{\alpha}\in\hat{\Delta}^{Re}_+}(1-e^{-\hat{\alpha}}) \prod_{\alpha\in\Delta_+} 1-q^{(\lambda+\rho,\alpha^\vee)}}$$

Delign – Cvitanovic series of AKM algebra

-

	$su(1)$	$su(2)$	$su(3)$	g_2	d_4	f_4	e_6	e_7	e_8
k	$-\frac{6}{5}$	$-\frac{4}{3}$	$-\frac{3}{2}$	$-\frac{5}{3}$	-2	$-\frac{5}{2}$	-3	-4	-6
c	$-\frac{22}{5}$	-6	-8	-10	-14	-20	-26	-38	-62
	Lee-Yang	admm	admm		Lagrangian				

$$k = -\frac{h^\vee}{6} - 1$$

- Unrefined vac. characters are known in closed form (Arakawa)

$$\left(D_q^{(2)} - 5(h^\vee+1)(h^\vee-1) E_4 \right) ch_0 = 0$$

$$\Rightarrow h_1 = -\frac{h^\vee}{6} \quad (= \text{int} \quad \text{when} \quad \mathfrak{g} = d_4 \ e_6 \ e_7 \ e_8)$$

Characters

- $su(2)_{k \in \mathbb{N}}$ 的 integrable : Di Francesco chap. 14
- $su(2)_{k=\frac{t}{u}}$ admissible : Di Francesco chap 18.
- $SU(N)_{k \geq 1}$ integrable : Di Francesco
- $ADE_{k=1}$ vacuum module
- $\widehat{so}(2r)_{k=1}$ integrable (4 个) } Di Francesco chap. 15
- $\widehat{so}(2r+1)_{k=1}$ integrable (3 个) }
- $\widehat{su}(N)_{k=-1}$ many modules : Adamovic and Milas
- $\widehat{so}(8)_{-2}$ vacuum module, and other from Category $\mathcal{O}$
- $\widehat{su}(N)_{k=-N}$: Arakawa.

Free field realization

		irrep R_λ
$\widehat{so}(N)_{k=x_\lambda}$	N independent real fermions	$J^a = \frac{1}{2}\psi_i t^a_{ij} \psi_j$
$\hat{u}(N)_{k=1}$	N independent cpx fermions	$J^a = \sum \psi_i^\dagger t^a_{ij} \psi_j$ $J^0 = \sum_i \psi_i^\dagger \psi_i$
$\widehat{su}(2)_{k=1}$	1 X	$H \sim i\partial X$ $E^\pm \sim e^{\pm\sqrt{2} i X}$
$\widehat{ade}_{k=1}$	$X_{I=1,\cdots,N}$	$H^I \sim i\partial X_I$ $E^\alpha \sim e^{\pm i \alpha^I H^I}$
$su(2)_{\forall k}$ Wakimoto	X, $\beta\gamma$.	$e \sim \beta$ $\vdots$
$\hat{\mathfrak{g}}_{\forall k}$ Wakimoto generalized	$\lvert\Delta_+\rvert$ 个 X rank 组 $\beta\gamma$	

- Recent: Adamovic / Bonetti - Meneghelli - Rastelli
  For 2d $\mathcal{N}=4$ small SCFA and W_G
  in terms of rank ↑ $\beta\gamma$
  $\Longrightarrow$ closed form characters for a lot of VOAs!

Beem - Meneghelli - Rastelli
DC series in terms of $\beta\gamma$ systems and X, Y

SUSY

- Extension of isometry (Poincare on flat spaces) by fermionic transformations (ϵ is Grassmann, spinor)

$$\delta\phi = \epsilon\psi \quad \text{or} \quad \delta A_\mu = (\epsilon\gamma_\mu\psi)$$

$$\delta\psi = \epsilon\phi + \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}\epsilon + \cdots$$

- P_μ, $M_{\mu\nu}$ Q_I ${R_I}^J$ $\supset$ SUSY on curved space.

- Superconformal algebra: extension of conformal algebra.

$$P_\mu, \quad M_{\mu\nu}, \; D, \quad K_\mu, \quad {R_I}^J, \quad Q_I, \; S^I,$$

4d $\mathcal{N}=2$

- Lagrangian theories building blocks : VM HM

G-VM : A, ϕ, $\tilde{\phi}$, λ_I, $\tilde{\lambda}_I$, D_{IJ} (in G-adj.)

(G-gauge field 升级版)

HM : q_{IA} ψ_A $\tilde{\psi}_A$ F_{IA} (in R) $I=1,2$ $A=1,2$

(matter field 升级版)

- SUSY param ϵ_I, $\tilde{\epsilon}_I$

$$\delta A_\mu = (\epsilon^I \sigma_\mu \lambda_I) + (\tilde{\epsilon}^I \sigma_\mu \tilde{\lambda}_I), \quad \delta\phi = \cdots, \quad \cdots$$

$$\delta q_{IA} = (\epsilon_I \psi_A) + (\tilde{\epsilon}_I \tilde{\psi}_A), \quad \delta\psi_A = \cdots, \quad \cdots$$

- $$\mathcal{L} = \frac{1}{2g_{YM}^2}\left(\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} - 4 D_\mu\phi D^\mu\tilde{\phi} + \cdots \right) + \theta\text{-term}$$

$$+ \frac{1}{2} D_\mu q^{IA} D^\mu q_{IA} + \cdots + \text{Yukawa 耦合}$$

$$+ \text{scalar potential} + \cdots$$

[Hosomichi 1206.6359]

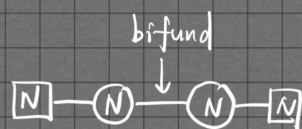
- P.F. $Z = \int D\Phi \, e^{-S}$

- 例:

$G = SU(N)$ $\quad N_f = 2N$ HM in fund : SQCD

$G = SU(N)$ $\quad$ 1 HM in adj. $= \mathcal{N}=2^* \xrightarrow{\text{massless}} \mathcal{N}=4$.

$G = SU(N)\times SU(N)$

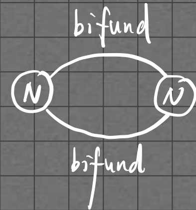


2 node linear quiver.

r node linear quiver

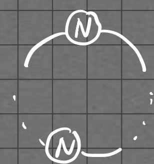
A_r quiver

$G = SU(2) \times SU(N)$



affine $\hat{A}_1$ quiver

with gauge group $SU(N)$

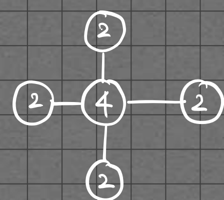


r+1 ↑ node

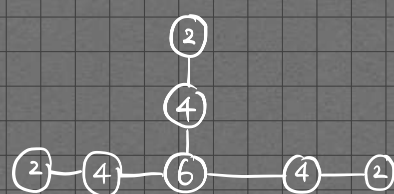
affine. $\hat{A}_r$ - quiver.

with gauge group $SU(N)$

Affine quivers



$\hat{D}_4\,[SU(4)]$



$\hat{E}_6\,[SU(6)]$ quiver

- N_f HM in $\begin{cases} \text{cpx} & SU(N_f) \times U(1) \\ \text{real} & USp(2N_f) = Sp(N_f) \\ \text{pseudo real} & SO(2N_f) \end{cases}$

- SU(2) − 4 flavors in $\underline{2}$: $\underline{2}$ 是 pseudoreal , $f = SO(8)$
- SU(3) − 6 flavors in $\underline{3}$: $\underline{3}$ 是 cpx , f : $U(6)$
- SU(2) − 1 flavor in adj : adj 是 real f: $USp(2) = Sp(1) = SU(2)$
- SU(N>2) − 1个 bifund : bifund 是 cpx $f = U(1)$

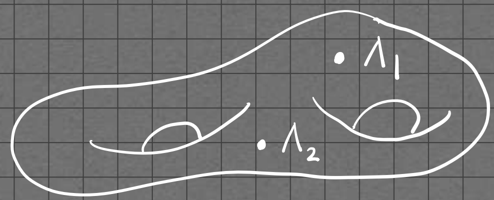
  SU(2) − 1个 bifund : pseudoreal $f = SU(2)$

class S

- twisted compactification from 6d (0,2) down to 4d
- Are 4d $\mathcal{N}=2$ SCFTs

李代数同态单射.

- Ingredients $\{\, \mathfrak{g} \in ADE \,,\ \Sigma_{g,n} \,,\ \Lambda_1, \cdots, \Lambda_n \mid \Lambda_i : su(2) \to \mathfrak{g} \,\}$,



- Puncture flavor f_i = $[\, f_i \,,\ \Lambda_i(su(2)) \,] = 0$

Manifest flavor $\oplus f_i$

- For $\mathfrak{g} = A_{N-1} = su(N)$,

$\forall \Lambda : su(2) \to \mathfrak{g}$ 可等价地看成 N 的配分

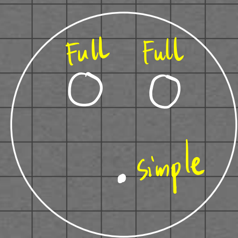
$[\, n_1^{l_1} \, n_2^{l_2} \cdots \,]$ s.t. $\sum_i l_i n_i = N$. , $n_i > n_{i+1}$

- full / maximal = $\mathrm{Im}\,\Lambda = 0$ (trivial) $\Leftrightarrow$ $[1^N]$ $\Leftrightarrow$ $f = \mathfrak{g}$

no puncture = principal embedding $\Leftrightarrow$ $[N]$ $\Leftrightarrow$ $f = \{0\}$

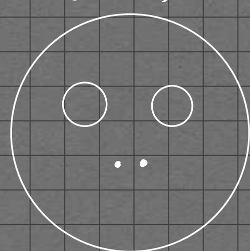
minimal / simple = subregular embedding $\Leftrightarrow$ $[N-1, 1]$ $\Leftrightarrow$ $f = u(1)$

- 通常是 strongly coupled. contains op with fractional dimensional
- some have Lagrangian descriptions.
- $(A_{N-1}, \Sigma_{0,3};\ \text{full}, \text{full}, \text{simple}) = N^2$ free HM



$$SU(N)^2 \times U(1) \subset USp(2N^2)$$

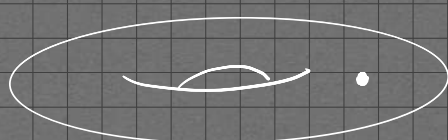
- $(A_{N-1}, \Sigma_{0,3};\ \text{full}, \text{full}, \text{simple}, \text{simple}) = SU(N)$ SQCD (2N HM in fund)



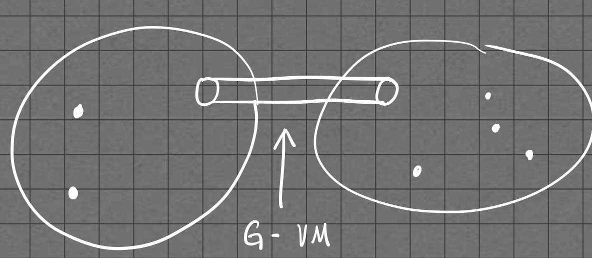
$$SU(N)^2 \times U(1)^2 \subset U(2N)$$

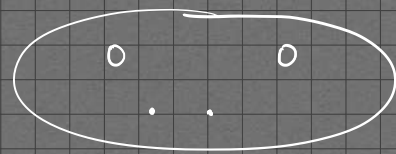
- $(A_{N-1}, \Sigma_{1,1};\ \text{simple}) = SU(N)$ $\mathcal{N}=4$ theory + 1 free HM.

SU(2)

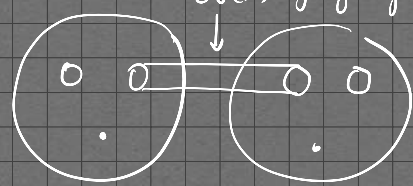


- gauging 2 G-symm: glue two maximal punctures.

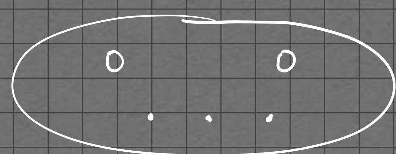


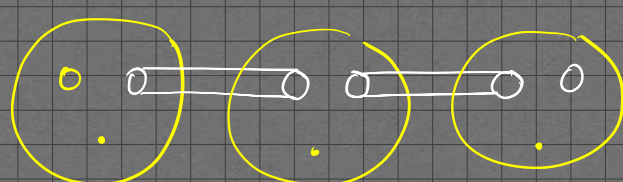
· SQCD =  = $\boxed{N} - N - \boxed{N}$

SU(N) gauge group / VM

=  = gauging two N^2 free HM.

N^2 free HM N^2 free HM



= 

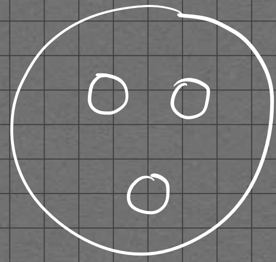
N^2 free HM N^2 free HM N^2 free HM

pants decomposition (Not unique) (related by generalized S-duality)

= $\boxed{N} - N - N - \boxed{N}$

2-node linear quiver
(A_2-quiver)

- $T_{N\geq 3}$ theories (3 maximal ; A_{N-1})



non - Lagrangian, strongly coupled.

S-duality

The Teichmuller space.

- $\mathcal{M}_{g,n}$: moduli space of cpx stru. of Riemann surface $\Sigma_{g,n}$
- $\partial\mathcal{M}_{g,n} \neq \phi$ 当 cpx str. 取 $\partial\mathcal{M}_{g,n}$ 时. $\Sigma_{g,n}$ 形成

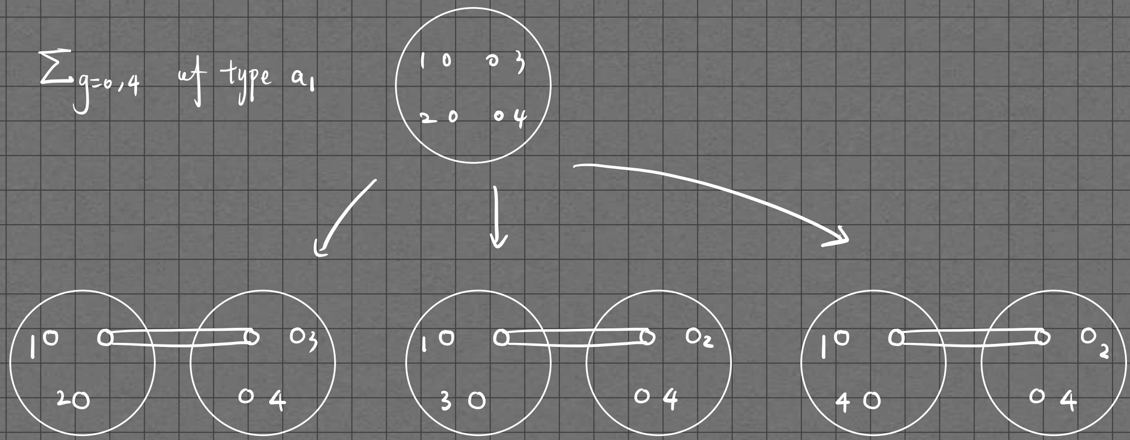
pants decompositions

- plumbing param of each cylinder $s \sim e^{2\pi i \underbrace{\left(\frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{YM}}\right)}_{\tau}} + \cdots$

The form local coordinates on $\mathcal{M}_{g,n}$

- Different pants decomposition / weak coupling limit.

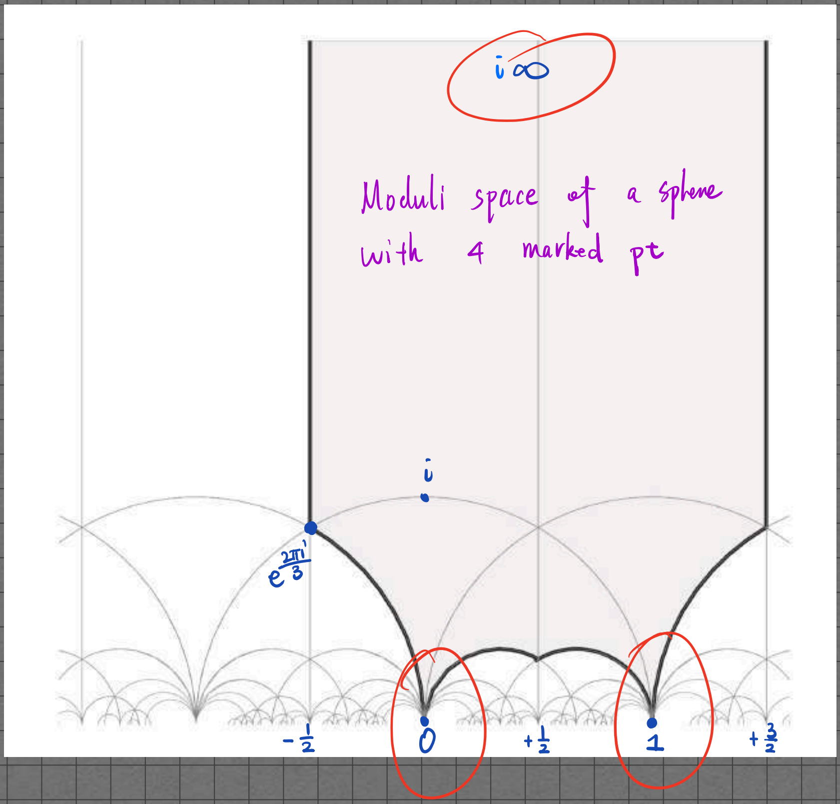
$\Sigma_{g=0,4}$ of type a_1



$\tau = i\infty$ $\tau = 0$ $\tau = 1$

3 S-duality frames / weak coupling limit.

- $\dim_{\mathbb{C}} \mathcal{M}_{g=0,4} = 1$: 参数为 τ.



Localization

- $Z^M \equiv \int D\Phi \, e^{-S_E[\Phi]}$ for a Lagrangian theory on M.

- When SUSY: assume $D\Phi$ is SUSY, $\delta D\Phi = 0$

  Choose gauge inv. fermionic V. s.t. $\delta^2 V = 0$, $\delta V|_B \geq 0$

$$\int D\Phi \, e^{-S - t\delta V} \Rightarrow \frac{d}{dt}\int D\Phi \, e^{-S-t\delta V} = 0$$

$t=0$ $\quad$ $t \to +\infty$

$$Z^M = \int_{\text{BPS}} e^{-S_{BPS}} \cdot \underbrace{\text{quadratic fluctuation}}_{Z_{pert}} \cdot Z_{inst/vortex}.$$

$(\delta V)_B = 0$ 的 field config

- M 常见 S^n $\quad S^n \times S^1$, $\quad \Sigma_{g,n} \times S^1$, $\quad \Sigma_{g,n}$

- 还可插入 BPS / almost BPS ops.

$$\langle \mathcal{O}(z_1, \cdots) \rangle = \int D\Phi \, \mathcal{O}(z_1, \cdots) \, e^{-S}$$

if $\frac{d}{dt}\int D\Phi \, \mathcal{O}(z_1,\cdots) e^{-S-t\delta V} = 0$, 则可用 localization

① $\delta\mathcal{O} = 0$ $\quad$ BPS

② $\delta\mathcal{O}$ = auxiliary fields $\quad$ almost BPS.

$S^3 \times S^1$ localization

- $$g = \cos^2\theta\left(d\varphi - \frac{i}{4\pi}(\beta_1+\beta_2)dt\right)^2 + \sin^2\theta\left(d\chi - \frac{i}{4\pi}(\beta_1-\beta_2)dt\right)^2$$
$$+ d\theta^2 + \left(-i\tau + \frac{\ell}{4\pi}(\beta_1+\beta_2)\right)^2 dt^2$$

- $$A^{SU(2)_R} = -\frac{1}{2}\left(\tau + \frac{i}{2\pi}(\beta_1+\beta_2)\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_I{}^J \quad \cdots$$

- $$D_\mu \xi_I = -i\sigma_\mu \tilde{\xi}'_I \qquad D_\mu \tilde{\xi}_I = -i\tilde{\sigma}_\mu \xi'_I$$

$$D_\mu \xi_I = \partial_\mu \xi_I + \frac{1}{4}\omega_\mu{}^a{}_b \sigma_{ab}\xi_I - i(A_\mu^{SU(2)_R})_I{}^J \xi_J + i A_\mu^{U(1)_r}\xi_I$$

$$\sigma^\mu \tilde{\sigma}^\nu D_\mu D_\nu \xi_I = M\xi_I \qquad \tilde{\sigma}^\mu \sigma^\nu D_\mu D_\nu \tilde{\xi}_I = M\tilde{\xi}_I$$

$\Rightarrow$ 4 solutions (c_i, $\tilde{c}_i$ 为任意数)

$$\xi_1 = c_1 K_{+-} \qquad \xi_2 = c_2 K_{++} \qquad \tilde{\xi}_1 = \tilde{c}_2 K_{--} \qquad \tilde{\xi}_2 = \tilde{c}_1 K_{-+}$$

- 定下 δ Fields

- 定下 $$\delta V = S_{YM} = \underbrace{\frac{1}{g_{YM}^2}\int_{S^3\times S^1} d^4x\sqrt{g}\,\mathcal{L}_{SYM} + \frac{i\theta}{8\pi^2}\int_{S^3\times S^1} \mathrm{tr}\, F\wedge F}_{\delta\text{ - exact}}.$$

$$\mathcal{L}_{YM}\big|_B = \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \overline{D_\mu\phi}\, D_\mu\phi + \cdots$$
$$\geq 0$$

$$\delta\mathcal{L}_{YM} = 0$$

- $$Z = \int D\Phi \; e^{-S_{YM} - S_{HM}}$$
$$= \int D\Phi \; e^{-S_{YM} - S_{HM} - (s-1)\underbrace{S_{YM}}_{\delta\text{-exact}}}$$

- BPS : $\underbrace{F_{\mu\nu}}_{\text{flat connection}} = 0 = \phi = \tilde{\phi} = D_{IJ}$,

$\Rightarrow \quad A = a\,dt \;, \quad a \in \mathfrak{h}$

q_{IA} 与 χ 无关. $\quad \underbrace{D_{t,\varphi,\theta}}_{\text{elliptic op.}} q = 0 \quad$ on $\quad S^1_t \times S^3_{\varphi,\chi,\theta} / U(1)_\chi$

$= S^1_t \times D^2_{\varphi,\theta}$

$\Rightarrow$ solutions determined by boundary values Q, $\tilde{Q}$ on $T^2_{t,\varphi}$

- $$S|_{BPS} = \int d\varphi\, dt \, (Q D_{\bar{z}} \tilde{Q} - \tilde{Q} D_z Q) \quad \sim \text{ symplectic boson.}$$
$\beta\gamma$ at $h = \frac{1}{2}$
$$= S_{SB}[Q, \tilde{Q} ; a]$$

- $$Z = \int da \;\; DQ D\tilde{Q} \;\; e^{-S_{SB}[Q,\tilde{Q};a]} \; \underbrace{Z_{1\text{-loop}}}$$
$$= \int Db\, Dc \; e^{-S_{bc}[b,c;a]}$$
$$= \int da \; DQ D\tilde{Q} \; Db\, Dc \;\; e^{-S_{SB}[Q,\tilde{Q};a] - S_{bc}[b,c;a]}$$

- Further insertions of special ops.

$$\langle \mathcal{O}_{bc\beta\gamma}(z_1, \cdots) \rangle_{S^3 \times S^1} = \int da \; \underbrace{\langle \mathcal{O}_{bc\beta\gamma}(z_1, \cdots) \rangle^{bc\beta\gamma}_{T^2_{\tau,\varphi}}}_{\text{2d VOA correlators on } T^2}.$$

- 4d $\mathcal{N}=2$ SCFT / 2d VOA

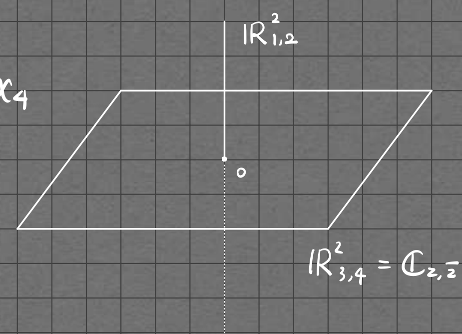
4d / 2d correspondence

- 考虑 ∀ 4d $\mathcal{N}=2$ SCFT.

- SCFA $D,\ P_\mu,\ M_\alpha{}^\beta,\ M_{\dot\alpha}{}^{\dot\beta}\ K_\mu,\ R^I{}_J$ $\qquad Q^I_\alpha \quad \tilde{Q}_{I\dot\alpha} \quad S_I^\alpha \quad \tilde{S}^{I\dot\alpha}$ $\quad \dagger \quad \dagger$

  quantum #: $E,\ j_1,\ j_2,\ R,\ r$

- Pick $\mathbb{R}^2_{3,4} \subset \mathbb{R}^4$. $\qquad z = x_3 + i x_4$



- Special subalgebra.

  $\{ L_{\pm,0} \}$

  $\{ \hat{L}_{\pm,0},\ Q^I_-,\ \tilde{Q}_{I\dot -},\ S_I^-,\ \tilde{S}^{I\dot -},\ \}$

- Special supercharges $\quad \mathbb{Q}_1 \equiv Q^1_- + \tilde{Q}_{2\dot -}^{\dagger}, \quad \mathbb{Q}_2 \equiv Q^{1\dagger}_- - \tilde{Q}_{2\dot -}$

  nilpotent

- $\mathbb{Q}_{1,2}$ - closed translation $\quad L_{-1}$

  $\mathbb{Q}_{1,2}$ - exact translation $\quad \hat{L}_{-1}$

- Special ops at origin: simultaneous cohomology of $\mathbb{Q}_{1,2}$

  Harmonic reps: $E-2R-j_1-j_2=0$ $\quad r-j_2+j_1=0$

- twisted translation on $\mathbb{C}_{z,\bar{z}}$ $e^{-\bar{z}\hat{L}_{-1}}\,e^{-zL_{-1}}$ to obtain $\mathcal{O}(z,\bar{z})$

  $[\mathcal{O}](z)$ 与 $\bar{z}$ 无关

- $$[\mathcal{O}_1](z)\,[\mathcal{O}_2](w)\sim\sum_h \underbrace{\frac{1}{(z-w)^{h_1+h_2-h}}}_{\text{全纯}}[\mathcal{O}_h](w)$$

- $[\mathcal{O}](z)$ 形成 Associated VOA

- Associated VOA.

- $SU(2)_R$ symm current $J^{\mu}_{I=1,\,J=1}$ $\longrightarrow$ stress tensor T in VOA

  $\longrightarrow$ Virasoro subalgebra with $c_{2d} = -12\, c_{4d}$

  flavor symm G moment map op $M_{I=1,J=1} \longrightarrow$ affine current j_z

  $\longrightarrow$ affine subalgebra $\hat{\mathfrak{g}}_{k_{2d}}$, $k_{2d} = -\frac{1}{2} k_{4d}$.

$\tilde{Q}_{1\dot{-}}$: neutral under all these

- 4d SCFI:

$$\mathcal{I}(p,q,t) \equiv \mathrm{str}\; e^{-\beta \tilde{\delta}_{1\dot{-}}}\, p^{\frac{\delta_{1+}}{2}}\, q^{\frac{E-2j_2-2R-r}{2}}\, t^{R+r}\, b^f$$

$\downarrow$ $t \to q$ (Schur limit)

$$\mathrm{str}\; e^{-\beta \tilde{\delta}_{1\dot{-}}}\, p^{\delta_{1+}}\, q^{\frac{E-2j_2+r}{2}}\, b^f$$

$\tilde{Q}_{1\dot{-}}$, Q_{1+} neutral under $E-2j_2+r$

$\Rightarrow$ independence of p $\Rightarrow$ set $p=0$

contrib. from Schur ops

$$\mathcal{I}(q) = \mathrm{str}_{\mathrm{Schur}}\; q^{\frac{1}{2}(E-2j_2+r)}\, b^f$$

$$= \mathrm{str}_{\mathrm{Schur}}\; q^{E-R}\, b^f$$

$$= \mathrm{str}_V\; q^{L_0}\, b^f$$

- Schur index

$$\mathcal{I}(q) \equiv q^{\frac{1}{2}c_{4d}}\, \mathrm{str}_{\mathrm{Schur}}\; q^{E-R}\, b^f = \mathrm{str}_V\; q^{L_0 - \frac{c_{2d}}{24}}\, b^f = \mathrm{ch}_V$$

- Macdonal limit

Free theory VOA

- $$q_{IA} = \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix}$$

- $Q = q_{11} + \bar{z}\, q_{21}$ $\qquad$ $\tilde{Q} = q_{12} + \bar{z}\, q_{22}$

$Q(z)\,\tilde{Q}(w) \sim \dfrac{1}{z-w}$ $\qquad$ $\tilde{Q}(z)\,Q(w) \sim \dfrac{-1}{z-w}$

$\Rightarrow$ $\beta\gamma$ system, $T = \frac{1}{2}(\beta\partial\gamma - \partial\beta\,\gamma)$, $h_\beta = h_\gamma = \frac{1}{2}$

- $\lambda_I \quad \tilde{\lambda}_I \longrightarrow$ $\lambda_z = \lambda_1 + \bar{z}\lambda_2 \sim \partial c$

$\tilde{\lambda}_z = \tilde{\lambda}_1 + \bar{z}\tilde{\lambda}_2 \sim b$

$\lambda_z(z)\,\tilde{\lambda}(w) \sim \partial_z \dfrac{1}{z-w}$, $T = b\partial c$, $h_b = 1$, $h_c = 0$

free VM $\Rightarrow$ bc ghost.

- build complex Lagrangian theories by gauging

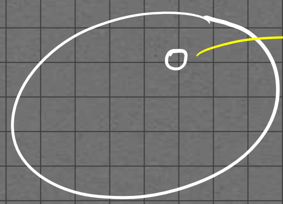
$\downarrow$

build complex VOA by BRST reduction of the product of free VOAs. $\mathrm{VOA}_1 \otimes \mathrm{VOA}_2$

$$Q_B = \oint c\,(\underset{\substack{\uparrow \\ \text{total flavor current to-be-gauged}}}{j_{tot}} + \frac{1}{2}\underbrace{j_{gh}}_{bc})$$

class S VOA

- $\mathfrak{g} \in A$



maximal G flavor subgroup.

$\downarrow$

$\hat{\mathfrak{g}}_{k=-h^\vee}$ subalgebra.

- building blocks : $T_{N\geq 3} \longrightarrow \mathrm{VOA}[T_N]$ is conjectured for A_{N-1}

$c_{2d} = -2N^3 + 3N^2 + N - 2$

$\widehat{su}(N)_{-N} \times \widehat{su}(N)_{-N} \times \widehat{su}(N)_{-N} \subset \mathrm{VOA}[T_N]$

额外 $W_{(l=2,3,\cdots N)}$ transforming in $\wedge^l \otimes \wedge^l \otimes \wedge^l$ under $su(N)^3$

额外 T (stress tensor)

- VOA [class S] from BRST reduction (gauging)

and "quantum Drinfeld-Sokolov reduction".

(reduce puncture)

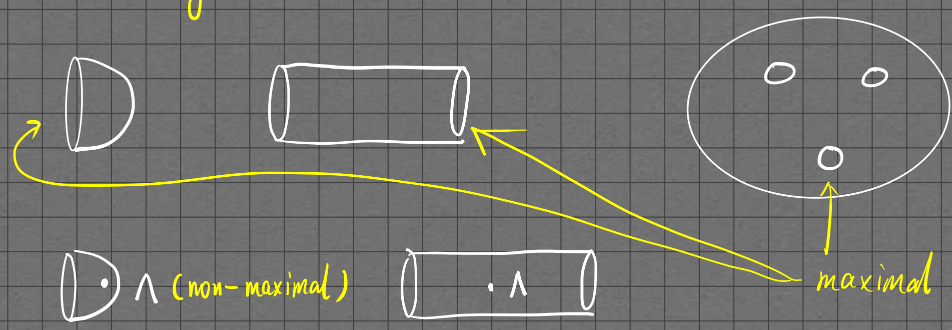
- Over all:

$$\begin{tikzcd}[column sep=small] & T[\Sigma_{g,n};\{\Lambda_i\}] \arrow[dr] & \\ (\Sigma_{g,n};\{\Lambda_i\}) \arrow[ur] \arrow[rr, "\mathbb{V}"] & & \mathrm{VOA}[T[\Sigma_{g,n};\{\Lambda_i\}]] \end{tikzcd}$$

Depend on the topology $(\Sigma_{g,n};\{\Lambda_i\})$ only (independent of gauge coupling)

$\Rightarrow$ $\mathbb{V}$ is a 2d TQFT valued in VOA

① basic building blocks (decorated Riemann surface)



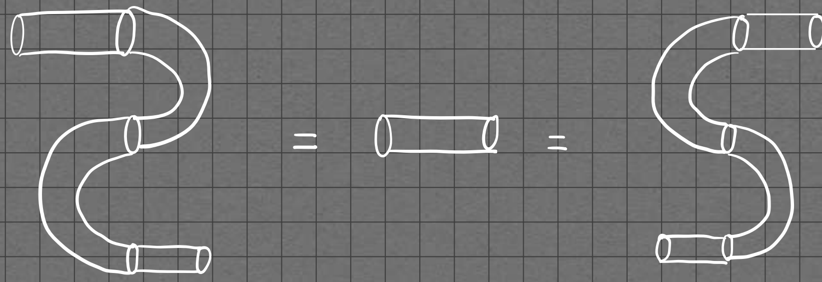
② each boundary / maximal puncture $\sim$ $\hat{\mathfrak{g}}_{-h^\vee}$

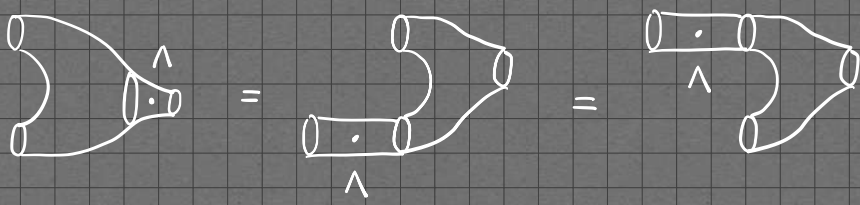
③ gluing along maximal punctures:

a) $\exists\,|\lambda$ (b,c) in the diag$(G\times G)$

b) BRST reduction

④ S - duality $\Leftrightarrow$ associativity





⋮

Schur indices

- $$\mathcal{I}_{HM} = \frac{\eta(\tau)}{\vartheta_4(\hat{b}|\tau)} \qquad b = e^{2\pi i \hat{b}}$$

- $$\mathcal{I}_{SU(2)\ \mathcal{N}=4} = \frac{1}{2}\oint_{|a|=1} \frac{da}{2\pi i a} \underbrace{\overset{\alpha}{\uparrow}\frac{\vartheta_1(2\hat{a})}{\eta(\tau)} \overset{-\alpha}{\uparrow}\frac{\vartheta_1(-2\hat{a})}{\eta(\tau)} \overset{0}{\uparrow}\eta(\tau)^2}_{\text{3 bc ghost charged under } su(2)_{gauge} \text{ adj.}} \underbrace{\frac{\eta(\tau)^3}{\vartheta_4(2\hat{a}+\hat{b})\vartheta_4(-2\hat{a}+\hat{b})\vartheta_4(\hat{b})}}_{\text{3 HM charged under } su(2)_{gauge} \text{ adj and } su(2)_f\ +1}$$

- $$\mathcal{I} = \frac{1}{|W|}\oint\left[\frac{da}{2\pi i a}\right] \eta(\tau)^{2r} \prod_{\alpha} \frac{\vartheta_1(\alpha(\hat{a}))}{\eta(\tau)} \prod_{R}\prod_{\rho\in R} \frac{\eta(\tau)}{\vartheta_4(\rho(\hat{a}))}$$

  for gauge theory.

- can be computed exactly in terms of Eisenstein series.

Schur indices from q-YM$_2$ (baby AGT)

- 1104.3850, 1408.6522

- $(\Sigma_{g,n}; \{\Lambda_i\}) \longrightarrow \mathcal{T}[\Sigma_{g,n}; \{\Lambda_i\}]$

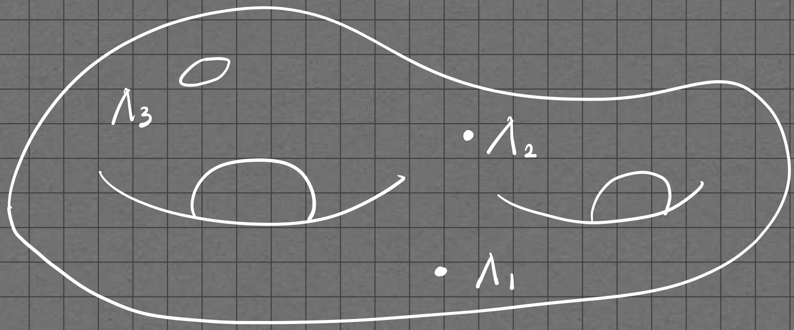
  $Z \quad = \quad \mathcal{I}$

- 2d G-YM: $S = \frac{1}{g_{YM}^2} \int \mathrm{tr}\, F \wedge F$

  $q \to 1 \uparrow \quad \downarrow$ "q-deformation" (at the level of amplitude)

  2d q-deformed G-YM

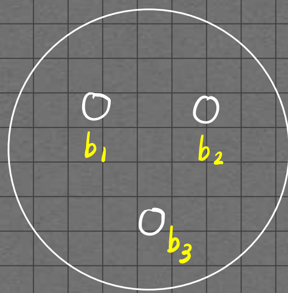
- $$A_{g,n} = q^{-\frac{C_{2d}}{24}} \sum_R C_R(q)^{2g-2+n} \prod_i \psi_R(x_i, \Lambda_i)$$
  $$= q^{-\frac{C_{2d}}{24}} \left\langle \prod_i \psi_R(x_i, \Lambda_i) \right\rangle$$
  $$\overset{\text{"AGT"}}{=} \mathcal{I}_{\text{Schur}}(\mathcal{T}[\mathfrak{g}; \Sigma_{g,n}; \{\Lambda_i\}])$$



- Simplest : 4 free HM

$\mathfrak{g} = su(2)$, 3 full

$SU(2)^3 \subset USp(8)$



$g = 0$

$n = 3$

$$\mathcal{I}_{Schur} = \prod_{\pm\,\pm} \frac{\eta(\tau)}{\vartheta_4(\hat{b}_1 \pm \hat{b}_2 + \hat{b}_3)}$$

$b_i \equiv e^{2\pi i \hat{b}_i}$

$$A = \sum_{R} C_R(q)^{2\cdot 0 - 2 + 3} \prod_{a=1}^{3} \psi_R(b_a ; \text{full})$$

$$= \sum_{j \in \frac{1}{2}\mathbb{N}} \frac{(q^2;q)}{\dim_q R_j} \prod_{a=1}^{3} \frac{\chi_{R_j}(b_a)}{(q;q)(b_a^2 q;q)(b_a^{-2} q;q)}$$

$$\dim_q R_j \equiv \frac{q^{\frac{1}{2}(2j+1)} - q^{-\frac{1}{2}(2j+1)}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \xrightarrow{q \to 1} 2j+1 = \dim R_j$$