Introduction to HEE

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量子引力与黑洞信息

References: RT, HRT, FLM, QES, …

Outline

- Basic introduction to AdS/CFT
- HRT
- FLM
- HRT v.s. Maximin Surface
- QES

Maximin Surfaces v.s.
Covariant Holographic Entanglement Entropy

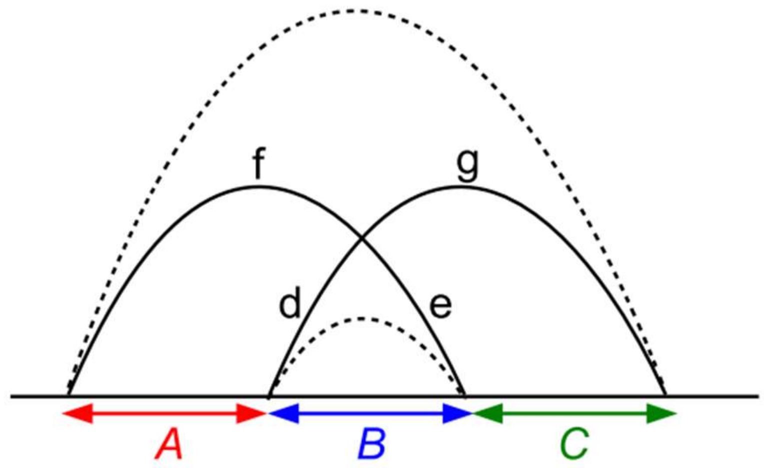
Introduction

Static Holographic Entropy

Ryu and Takayanagi

$$S_A = \frac{\mathrm{Area}[\min(A)]}{4\hbar G}.$$

$$S_{AB} + S_{BC} \geq S_{ABC} + S_B.$$



$$A[m(B)] \leq d + e$$

$$A[m(ABC)] \leq f + g$$

$$A[m(B)] + A[m(ABC)] \leq d + e + f + g$$

$$= A[m(AB)] + A[m(BC)]$$

Applies only to static manifolds

1.min[ABC] and min[B] are minimal surfaces
2.lying on the same time slice as min[AB] and min[BC].

Inapplicable on dynamically evolving spacetimes, or to choices of A which do not correspond to static time slices

Covariant Holographic Entanglement Entropy

Hubeny, Rangamani, and Takayanagi

Instead of looking for the minimal area surface, look for the extremal area surface m(A) (if there is more than one, choose the surface with the least area)

The matter obey the null energy condition

$$T_{ab}k^a k^b \geq 0$$

The null curvature condition (NCC)

$$R_{ab}k^a k^b \geq 0$$

Area[m(AB)] + Area[m(BC)] + Area[m(AC)] ≥ Area[m(A)] + Area[m(B)] + Area[m(C)] + Area[m(ABC)]

Maximin surface M(A): minimizing the area on some achronal slice Σ, and then maximizing the area with respect to varying Σ

Assumptions about Spacetime

The bulk spacetime： classical, smooth, and asymptotically locally AdS

NCC $R_{ab}k^a k^b \geq 0$ for any null vector k^a

Generic condition: nonvanishing null-curvature $R_{ab}k^a k^b$

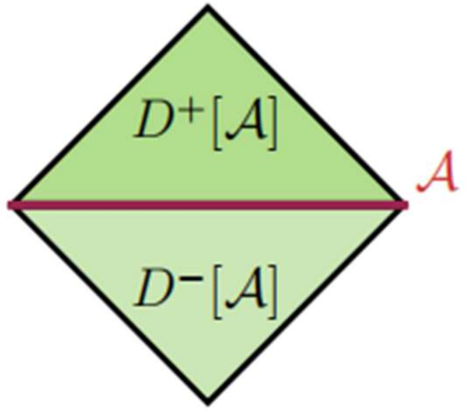
Or shear σab along at least one point of any segment of any null ray

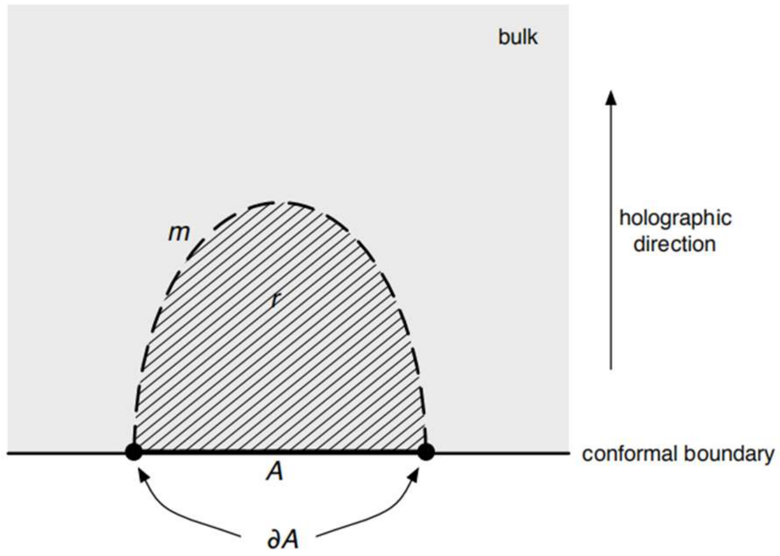
The spacetime will also be assumed to be AdS-hyperbolic

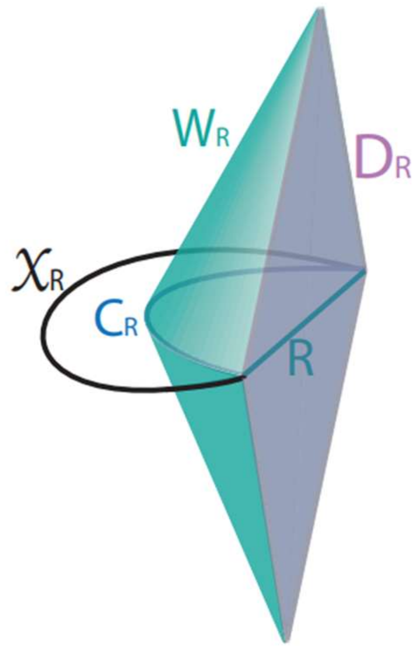
1. there are no closed causal curves,
2. for any two points x and y, I+(x) ∩ I -(y) is compact after conformally compactifying the AdS boundary

Space at one time is compact, after compactifying the AdS boundary

Definitions







$$D_A = D^+(A) \cup D^-(A)$$

causal wedge $\omega_A = I^-(D_A) \cap I^+(D_A)$

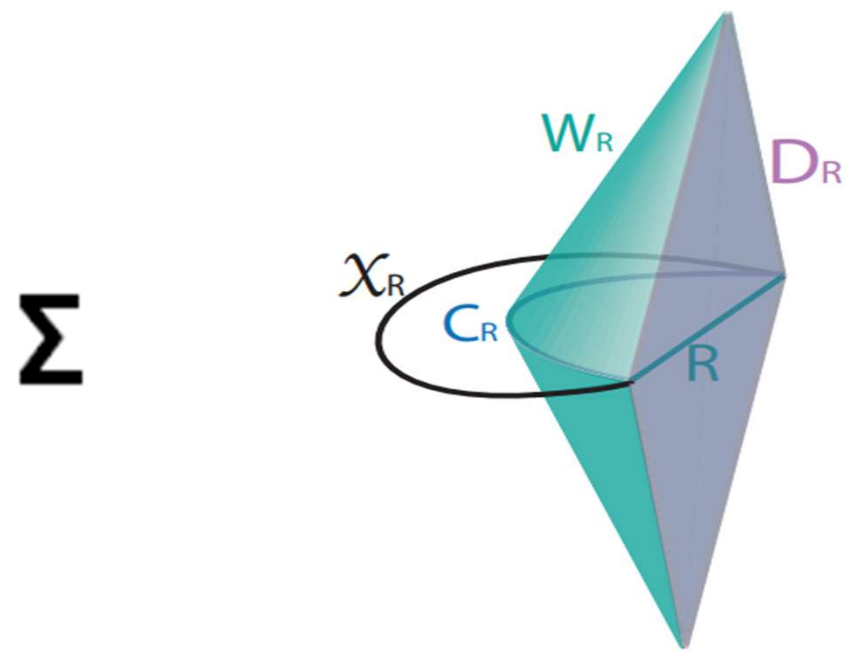
causal surface $C_A = \partial I^-(D_A) \cap \partial I^+(D_A)$

r(A) :the spacetime region lying spatially in between m(A) and DA

Preliminary Definitions and Lemmas

N(A): Codim 2 extremal surface x(A) 沿着光传播的方向构成的一个曲面Codim=1

Codim 2 “Representative” on Σ defined as $\tilde{x}(A, \Sigma) = N(A) \cap \Sigma$.



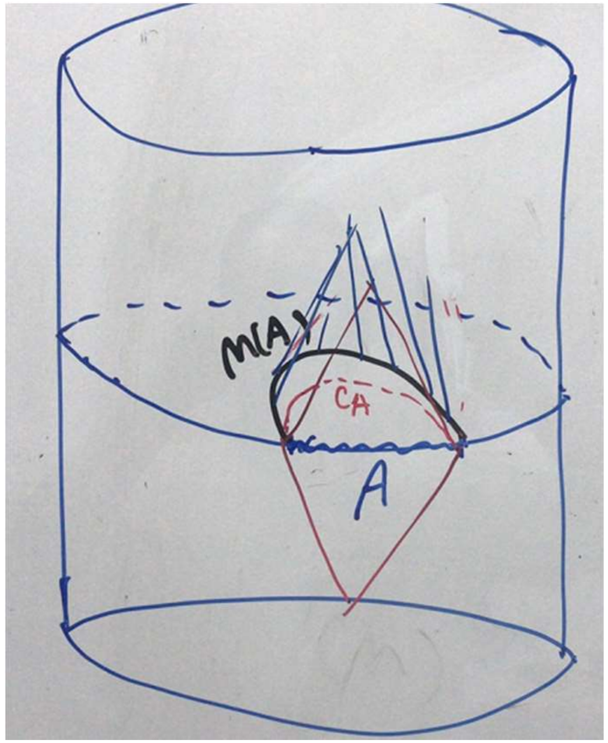
Theorem:
Representative ˜x(A, Σ) has less area than x(A) (unless it is x(A)).

① x(A) is extremal, the null surfaces N(A) have expansion θ = 0 at x(A)

②The Raychaudhuri equation

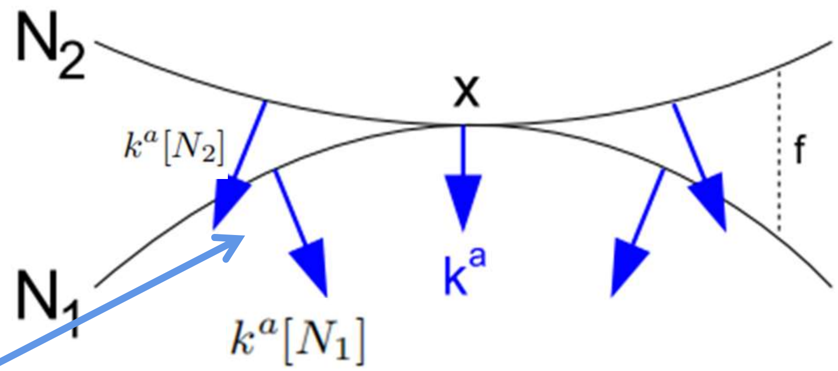
$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{D-2} - \sigma_{ab}\sigma^{ab} - R_{ab}k^a k^b,$$

③So that θ < 0 everywhere on N(A)



1. N1 and N2 be null congruences
2. N2 be nowhere to the past of N1

θ[N2] > θ[N1]



Null extrinsic curvature of a null surface

$$B_{ab} = h_a^c h_{bd} \nabla_c k^d,$$

Expansion $\theta \equiv (\text{Area})^{-1} k^a \nabla_a \text{Area} = B_{ab} h^{ab}.$

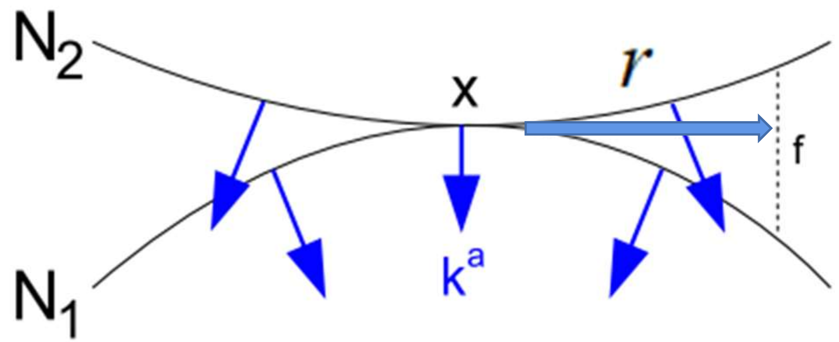
f在x处为零,在x邻域内领头阶为零，只有高阶的量

$$\Delta k^a = k^a[N_2] - k^a[N_1] = \nabla^a f + \mathcal{O}((\nabla f)^2).$$

$$\Delta B_{ab} = B_{ab}[N_2] - B_{ab}[N_1] = \nabla_a \nabla_b f.$$

$$\Delta \theta = \theta[N_2] - \theta[N_1] = \nabla^2 f,$$

$$-\nabla^2 G(y) = \delta^{D-2}(y); \qquad G|_{r=R} = 0,$$



Sufficiently small R,metric hab is very close to being a flat Euclidean metric

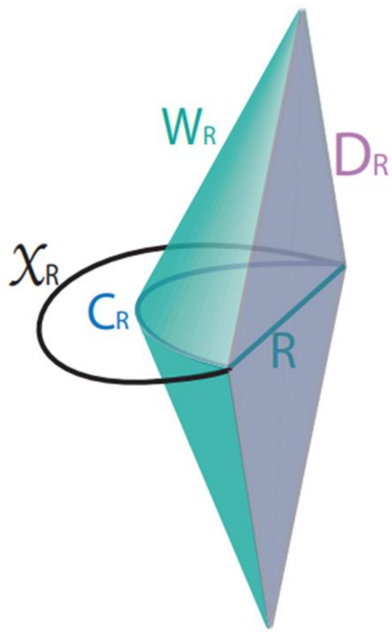
$$G \propto (r^{D-4} - R^{D-4})/(D-4)$$ **or** $\ln(R/r)$ **in D=4**

G(y) > 0 for r < R, and ∂rG|r=R < 0.

$$\int_B G\,\Delta\theta\, d^{D-2}y = \int_B G\nabla^2 f d^{D-2}y = -\int_{\partial B} f\,\partial_r G\, d\sigma \geq 0.$$

θ[N2] > θ[N1]
tr(K)[N2] >tr(K)[N1]

Extremal Surfaces lie outside Causal Surfaces



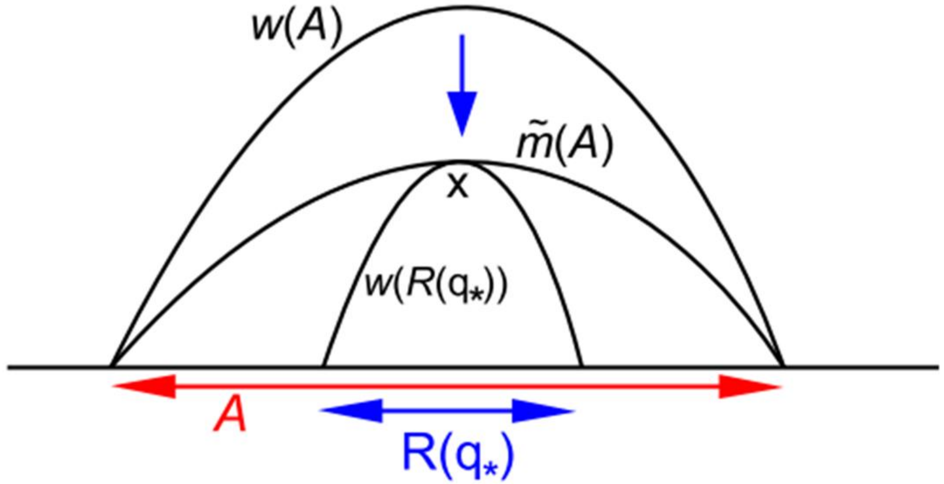
$$\omega(A) = \partial I^{-}(D_A) \cap \partial^{+}(D_A)$$

An extremal surface x(A) lies outside of w(A), in a spacelike direction

①w(R(q∗))has θ > 0 by the Second Law

N(A) has θ < 0 by Theorem 3.

N(A)<w(R(q∗))



②N(A) is nowhere inside w(R(q∗))

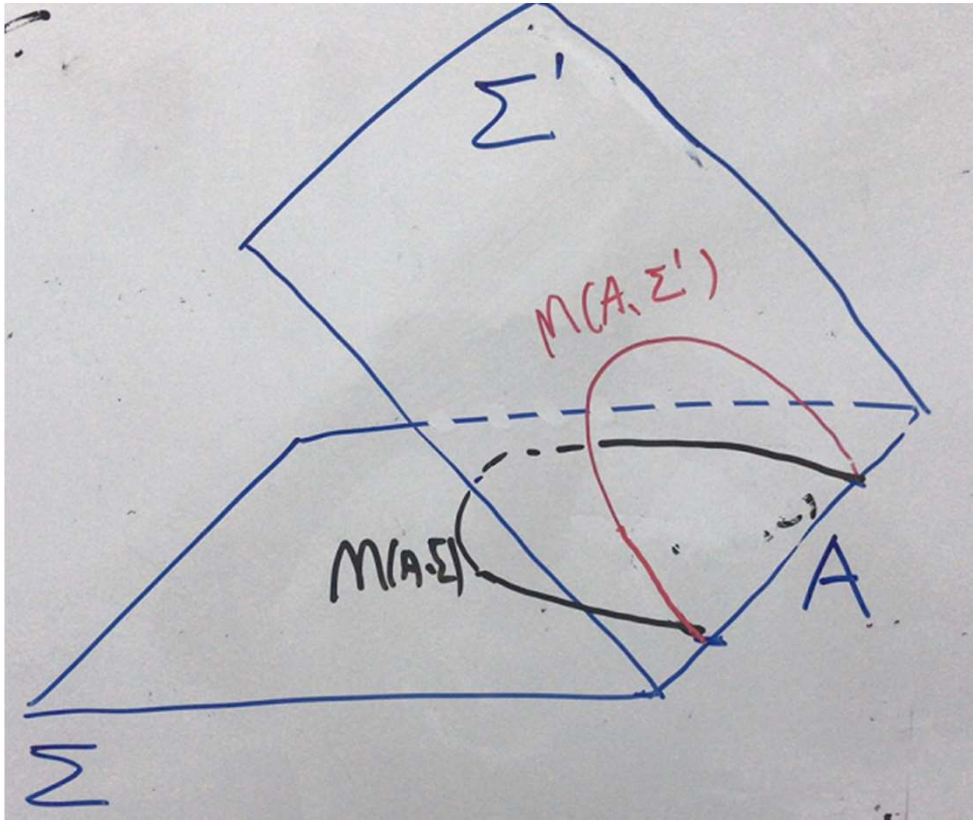
θ[N(A)] ≥ θ[w(R(q∗))]

Maxinmin Surface M(A)

Area[M(A)] ≥Area[M′(A)]

$\Sigma \rightarrow \Sigma'$

极大值



Equivalence of Maximin Surfaces and HRT Surfaces

M(A) is everywhere spacelike separated to itself

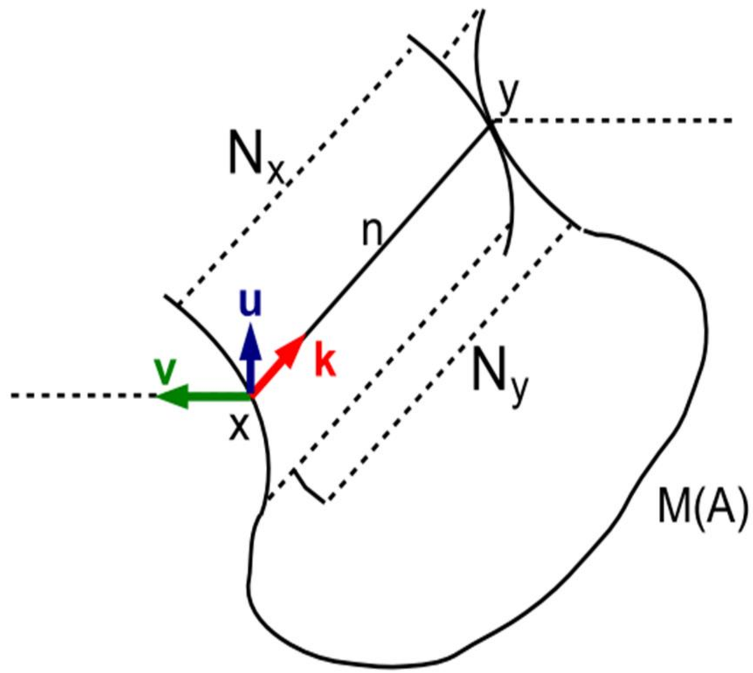
Neither can two points on M(A) be connected by a null segment n which does not lie on M(A)

M(A)是极值面 $\theta[M(A)] = 0$

x在M(A)上 **θ(Nx) = 0 at x**

沿光线n的切矢k方向，膨胀率θ下降，所以θ(Nx) < 0 at y

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{D-2} - \sigma_{ab}\sigma^{ab} - R_{ab}k^a k^b,$$



同理y在M(A)上，θ(Nx) = 0 at y,沿光线-k方向膨胀率上升，θ(Nx) > 0 at x，与x在M(A)上，θ(Nx) = 0 at x矛盾!

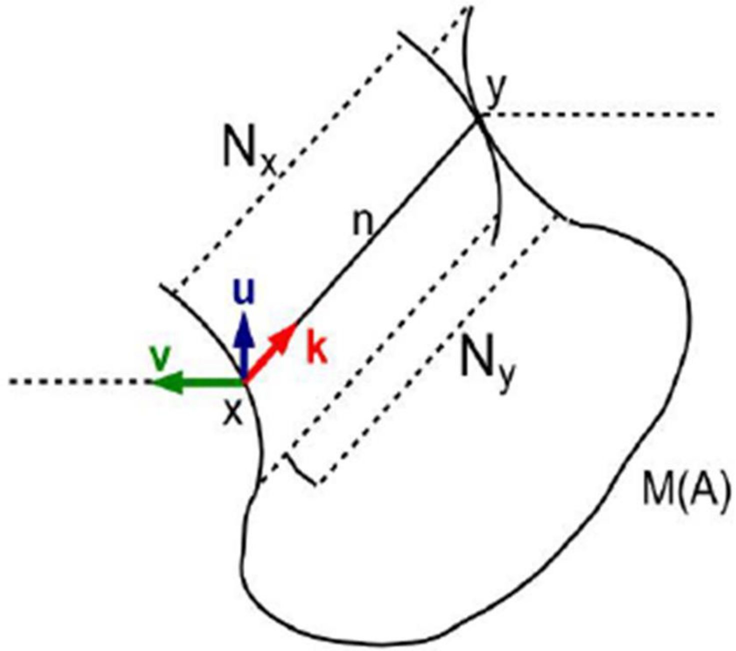
(c) Consider the trace of the extrinsic curvature at x

$$\mathrm{tr}(K)[M(A)] = (\mathrm{Area})^{-1}\nabla\mathrm{Area} \equiv K_i, \qquad \theta \equiv (\mathrm{Area})^{-1}k^a\nabla_a\mathrm{Area} = B_{ab}h^{ab}.$$

$\theta(N_x) = K_i k^i > 0$ （见上页PPT θ(Nx) > 0 at x的论述)

M(A) is minimal on Σ

v是Σ面上的一个切矢，M(A)在Σ上面积最小，因此沿v面积变大，沿v膨胀率θ大于0 $\rightarrow \theta = K_i v^i > 0$



(d) $\Sigma \rightarrow \Sigma'$

Area[min(A, Σ′)] ≤ Area[M(A)].

u是从Σ指向Σ'的向量，在所有Σ中，M(A)的面积最大，因此沿u面积变小，膨胀率θ小于0 $\rightarrow K_i u^i \leq 0$

两者矛盾

然而u可以写成k和v的正系数的线性组合 $\rightarrow K_i u^i > 0$

平行于Sigma

Consider the trace of the extrinsic curvature at x

$$\mathrm{tr}(K)[M(A)] = (\mathrm{Area})^{-1}\nabla \mathrm{Area} \equiv K_i, \qquad \theta \equiv (\mathrm{Area})^{-1} k^a \nabla_a \mathrm{Area} = B_{ab}h^{ab}.$$

M(A) is minimal on Σ

v是Σ面上的一个切矢，M(A)在Σ上面积最小，
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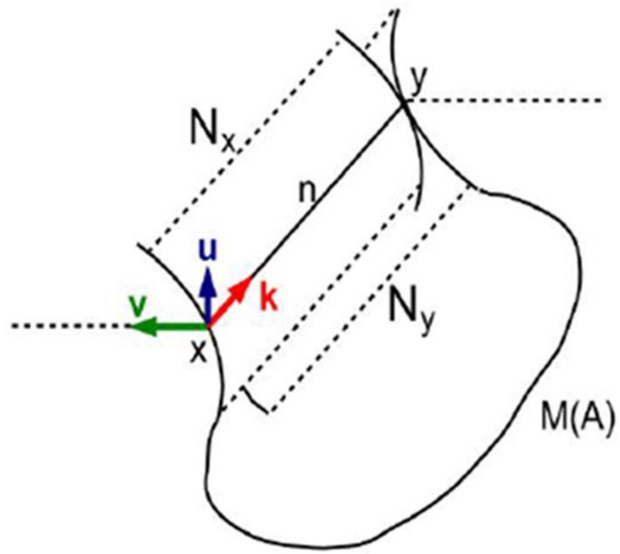


垂直于Sigma

$\Sigma \rightarrow \Sigma'$

Area[min(A, Σ')] ≤ Area[M(A)].

u是从Σ指向Σ'的向量，在所有Σ中，M(A)的面
积最大，因此沿u面积变小，膨胀率θ小于0 $\rightarrow K_i u^i \leq 0$

两者矛盾

然而u可以写成k和v的正系数的线性组合 $\rightarrow K_i u^i > 0$

Maximal+Minimal ⇒ **Extremal**

M(A) has less area than any extremal surface x(A) which does not lie on Σ.

$$\text{Area}[M(A)] \leq \text{Area}[\tilde{x}(A)] \leq \text{Area}[x(A)]$$

M(A) is minimal on Σ

Raychaudhuri equation

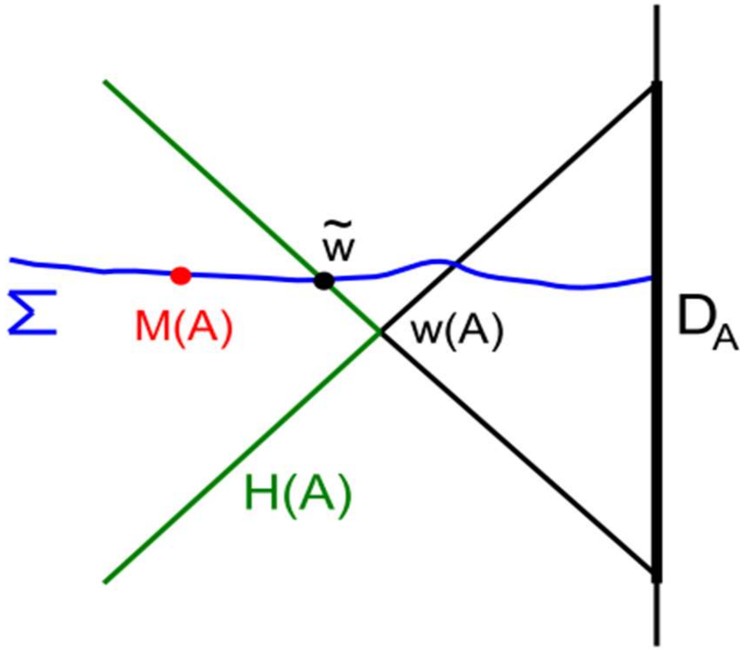
Maximal+Minimal → Extremal

Maximin Surface ⟺ HRT Surface

Properties of Maximin/HRT Surfaces

Less Area than the Causal Surface

$$\mathrm{Area}[w(A)] > \mathrm{Area}[\tilde{w}(A, \Sigma)] > \mathrm{Area}[m(A)]$$



By the Second Law of horizons,the area of H(A) decreases when moving away from w(A).

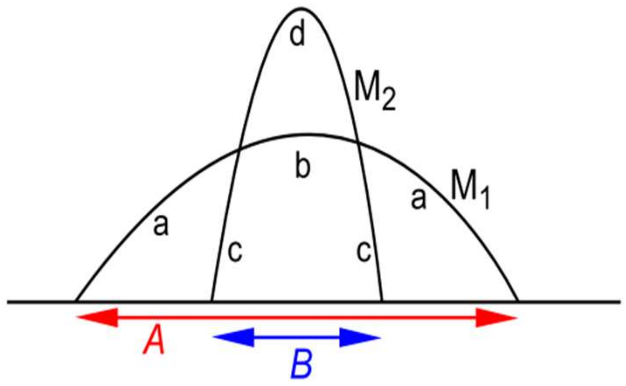
Moves outwards as the Boundary Region Grows

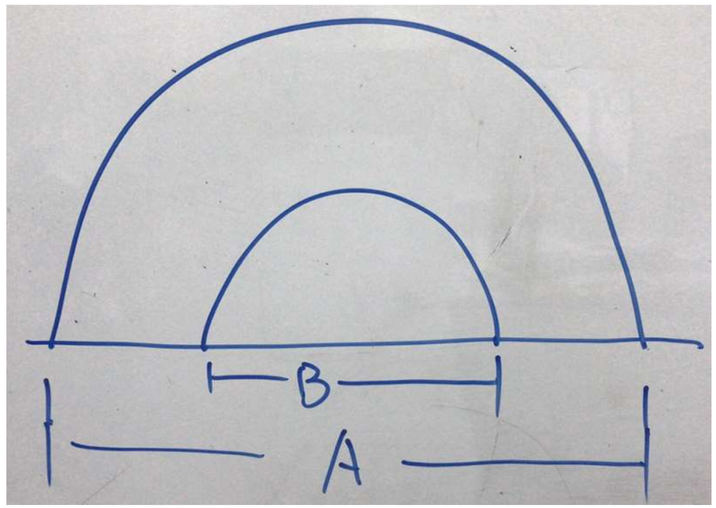
If A ⊃ B, then r(A) ⊃ r(B), with m(A) spacelike to m(B).

①M1 must lie spatially outside or on M2.

If Area[b] < Area[d], then Area[bc] < Area[cd] which contradicts the minimalness of M2.

If Area[b] > Area[d], then Area[ad] < Area[ab] which contradicts the minimalness of M1.





正确图像

②M1 and M2 cannot exactly coincide, because they are anchored to different points on the boundary tr(K) = 0

Strong Subadditivity

(a) Exists a spacelike slice Σ on which both m(ABC) and m(B) lie as minimal surfaces, with m(ABC) everywhere outside of m(B).

(b)Area[˜m(AB, Σ)]+Area[˜m(BC,Σ)]≥Area[m(ABC)]+Area[m(B)],

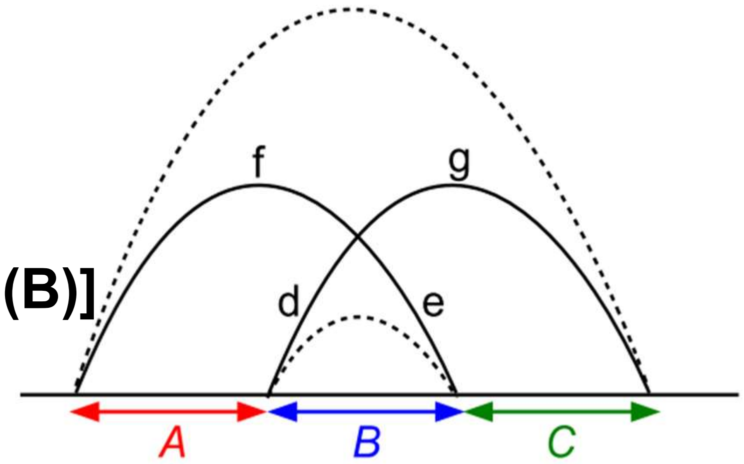
① 同一Σ面上的次可加性

(c)Area[m(AB)] + Area[m(BC)] ≥Area[˜m(AB, Σ)]+Area[˜m(BC,Σ)]

② 由Maximal Minimal

联立①②

Area[m(AB)] + Area[m(BC)] ≥ Area[m(ABC)] + Area[m(B)]

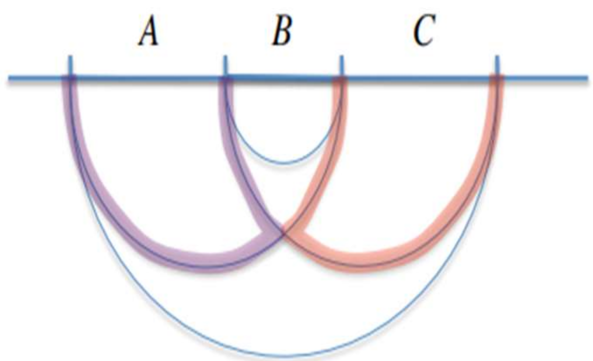


Monogamy of Mutual Information

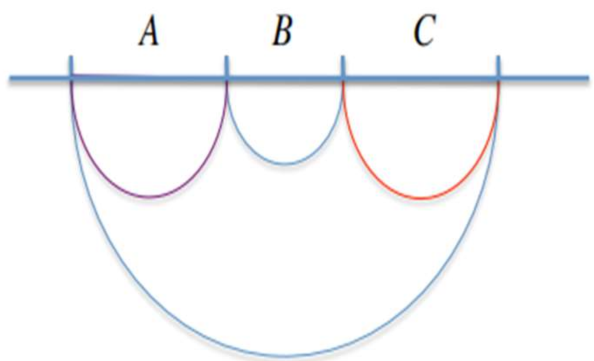
Area[m(AB)] + Area[m(BC)] + Area[m(AC)] ≥
Area[~m(AB)] + Area[~m(BC)] + Area[~m(AC)]≥
Area[m(A)] + Area[m(B)] + Area[m(C)] + Area[m(ABC)]

第一个不等号由引理3：
Area[m(AB)] ≥Area[~m(AB)] 可得

第二个不等号由在同一Σ面上的相互信息的性质可以得到



i) $S(AB)+S(BC)+S(AC)$



ii) $S(A)+S(B)+S(C)+S(ABC)$

Area[m(AB)] + Area[m(BC)] + Area[m(AC)] ≥
Area[m(A)] + Area[m(B)] + Area[m(C)] + Area[m(ABC)

Quantum Extremal Surface

Holographic EE

HRT entanglement entropy of region R $\longleftarrow$ extremal surface X

X is spacelike, codimension 2, extremal surface, $\partial X = \partial R$, X is homologous to R

$S(R)=A(X)/4G\hbar \qquad \mathcal{O}(1/\hbar)$

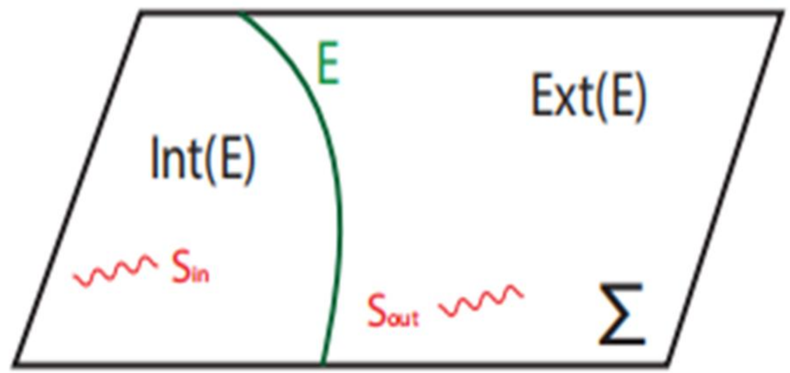
FLM $S_R = \frac{\langle A(X) \rangle}{4G\hbar} + S_{ent} + counterterms$ $\quad S_{ent}$ is the bulk entanglement entropy across X

$\mathcal{O}(\hbar^0)$

$S_R = S_{gen}(X)$

E spacelike codimension 2 surface

Σ Cauchy surface



$$S_{gen}(H) = \frac{\langle A(H) \rangle}{4G\hbar} + S_{out} + \text{counterterms}$$

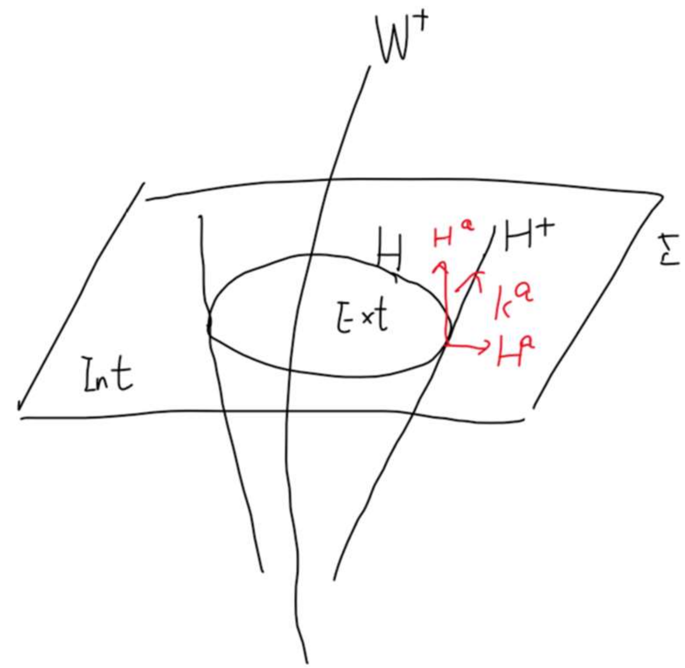
Pure $S_{in}(E) = S_{out}(E)$

Mixed Same side as the region R

(a) extremize the area and then add S_{out} FLM

(b) extremize the total generalized entropy S_{gen}

Definitions



Timelike or null worldline W^+ observer
Future causal horizon $H^+ = \partial I^-(W^+)$
Cauchy surface Σ
Horizon slice H=$\Sigma \cap H^+$

GSL

$$\frac{\delta S_{\text{gen}}(H)}{\delta H^a} k^a \geq 0$$

H^a is a normal vector field living on the surface H

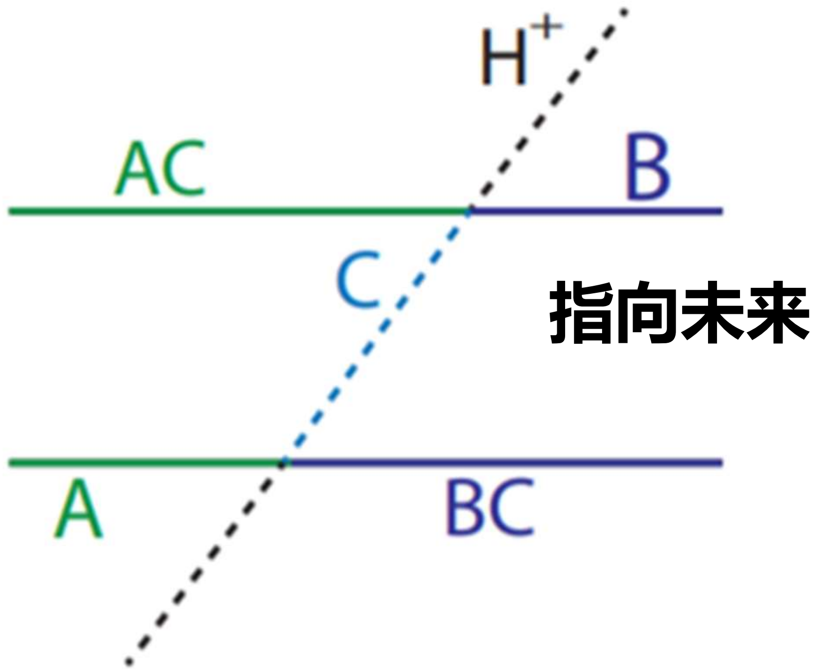
k^a is a future-pointing null vector parallel to the null generators of H$^+$

Storng subadditivity

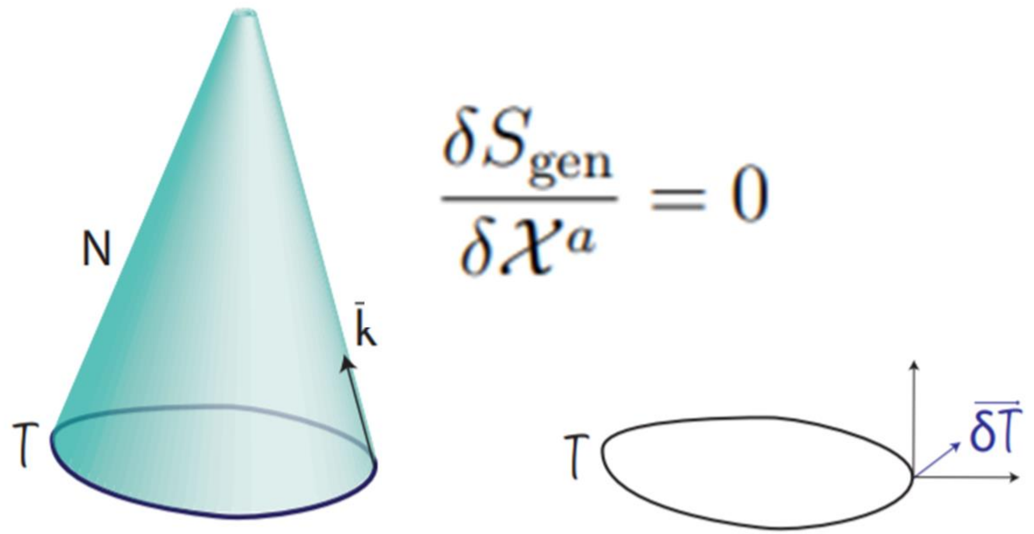
$$S(AC) + S(BC) \geq S(A) + S(B)$$

$$\frac{\delta S_{\rm in}(H)}{\delta H^a} k^a \geq \frac{\delta S_{\rm out}(H)}{\delta H^a} k^a$$

Future causal horizon $H^+ = \partial I^-(W^+)$



Spacelike, codimension 2 surface on Cauchy surface



(a) (b)

$$\frac{\delta S_{\text{gen}}}{\delta \mathcal{X}^a} = 0$$

$$\frac{\delta S_{\text{gen}}}{\delta \mathcal{T}^a} k^a < 0$$

Quantum trapped surface $\mathcal{T}$

Conjecture: The entanglement entropy of a region R in a field theory with a holographic dual is given at any order in $\hbar$ in the holographic dual by the generalized entropy of the quantum extremal surface $\mathcal{X}_R$ anchored at R and homologous to R:

$$S_R = S_{\text{gen}}(\mathcal{X}_R) \tag{3.1}$$

Comparison to Faulkner–Lewkowycz–Maldacena Formula

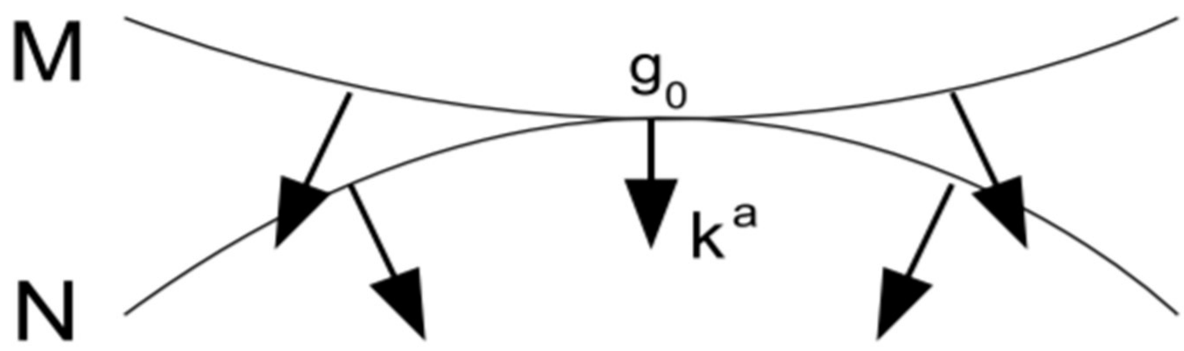
$$S_{\text{gen}}(X_R) = S_{\text{gen}}(\mathcal{X}_R) + \mathcal{O}(\hbar^1)$$

$$A(X_R) - A(\mathcal{X}_R) = \mathcal{O}(\hbar^2).$$

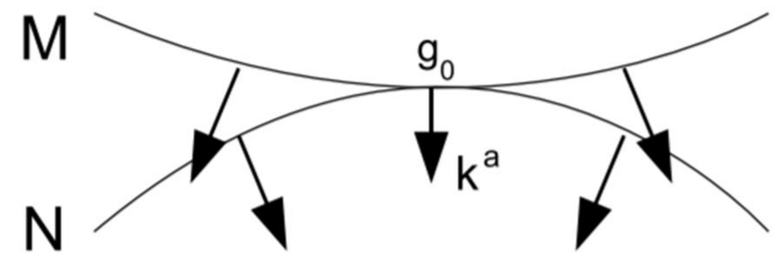
Theorem 2.1. *(from [27]): Let M, N be null splitting surfaces (i.e codimension 1 surfaces which divide spacetime into two regions [26] and have an open exterior) which coincide at a point p and let Σ be a spacelike slice that goes through p. If (1) $M \cap Ext(N) = \emptyset$, and (2) M, N are smooth at the classical order near p, and the spacetime is described by an $\hbar$ expansion there, then there exists a way of evolving Σ forward in time in a neighborhood of p so that*

$$\Delta S_{\text{gen}}(M) \geq \Delta S_{\text{gen}}(N) \tag{2.10}$$

with equality only if M and N coincide at a neighborhood.



Lemma B: In any small neighborhood of g_0, either there is a point X at which $\theta^{(M)} > \theta^{(N)}$, or else M and N coincide everywhere in that neighborhood. In the former case, the area increases faster on M than N when Σ is pushed forwards in time sufficiently close to the point X; in the latter case, the area increase is the same for M and N in the whole neighborhood. Either way, Theorem 1 holds classically.



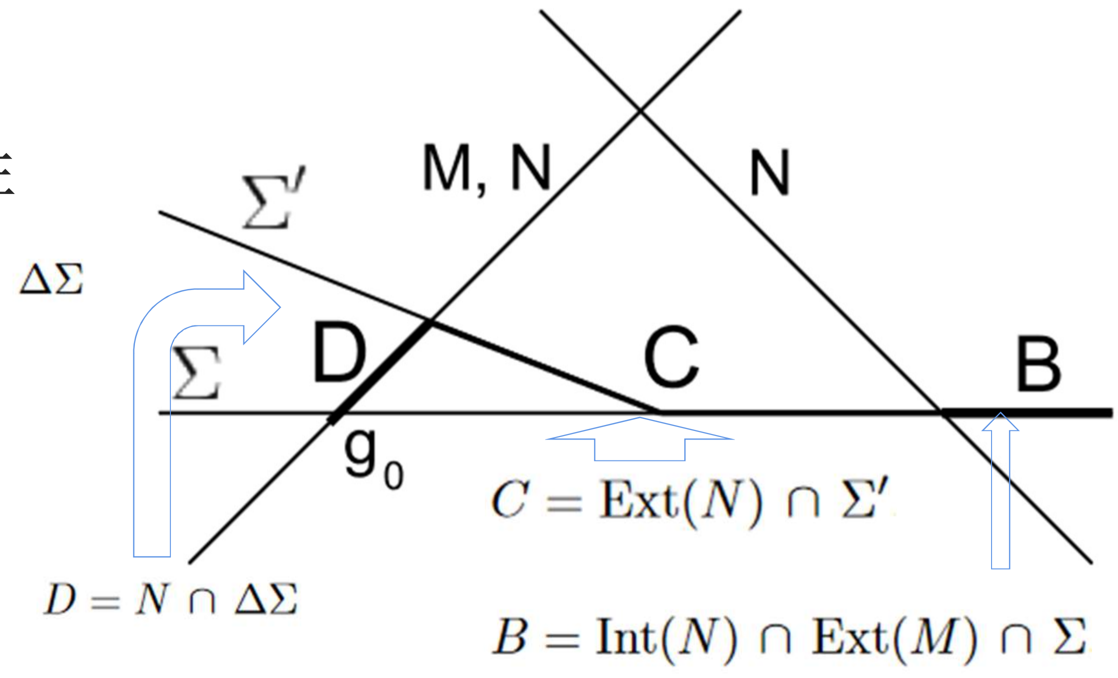
Lemma C: If the two surfaces N and M coincide in a neighborhood of g_0, and Σ is evolved forwards in time to Σ' in this neighborhood, the entropy S_{out} is increasing faster on M than on N.

$$I(B,C)=S(B)+S(C)-S(B\cup C).$$

互信息的单调性

$$I(B,C\cup D)\geq I(B,C).$$

定义：



$$I(B,C)=S(B)+S(\mathrm{Ext}(N)\cap\Sigma')-S(\mathrm{Ext}(M)\cap\Sigma').$$

$$I(B,C\cup D)=S(B)+S(\mathrm{Ext}(N)\cap\Sigma)-S(\mathrm{Ext}(M)\cap\Sigma),$$

$$\Delta S_{\mathrm{out}}(M)\geq\Delta S_{\mathrm{out}}(N),$$

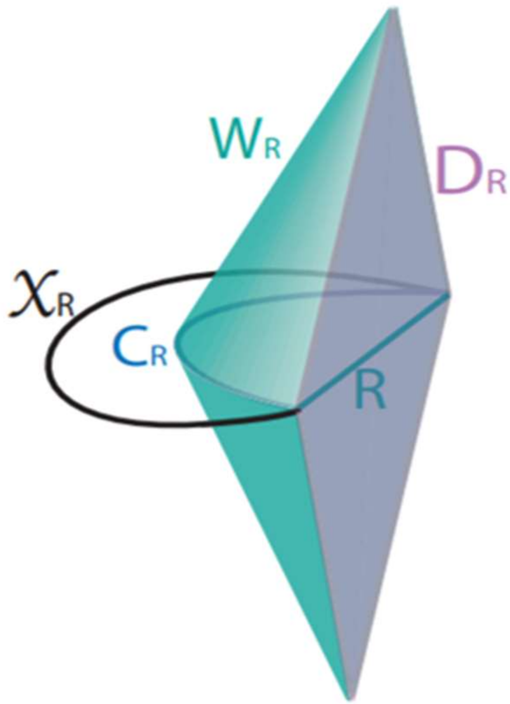
Proof of Theorem 1: In the semiclassical limit, any effect which is higher order in $\hbar$ will be dominated by any nonzero effect which is lower order in $\hbar$. Let the leading order contribution to $\theta^{(M)} - \theta^{(N)}$ be of order $\hbar^{p+1}$, which corresponds to an order $\hbar^p$ contribution to $\Delta S_{\mathrm{H}}^{(M)} - \Delta S_{\mathrm{H}}^{(N)}$, since the Bekenstein-Hawking entropy (9) has an $\hbar$ in the denominator. Lemma B says that at every order in $\hbar$, either N and M coincide or else $\Delta S_{\mathrm{H}}^{(M)} - \Delta S_{\mathrm{H}}^{(N)} > 0$ for an appropriate choice of Σ evolution. By applying Lemma B to order $\hbar^{p+1}$, one obtains that the order $\hbar^p$ contribution to $\Delta S_{\mathrm{H}}^{(M)} - \Delta S_{\mathrm{H}}^{(N)}$ is positive. By applying Lemma B at order $\hbar^p$, one obtains that N and M coincide at order $\hbar^p$. Since Q is subleading, there is no $\hbar^p$ order contribution coming from $\Delta Q^{(M)} - \Delta Q^{(N)}$.

Let the leading order contribution to $\Delta S_{\mathrm{out}}^{(M)} - \Delta S_{\mathrm{out}}^{(N)}$ be of order $\hbar^q$. If $p \leq q$, then the area term dominates over the entropy term. If $p \geq q$, then since at this order the null surfaces coincide, Lemma C says that the S_{out} increases faster for M than N. Either way, Theorem 1 follows.

$$S_{\mathrm{gen}} = \frac{A}{4\hbar G} + Q + S_{\mathrm{out}},$$

Quantum extremal surfaces lie deeper than causal surfaces

W_R	bulk causal wedge
C_R	causal surface
D_R	domain of dependence
$\mathcal{X}_R$	quantum extremal surface

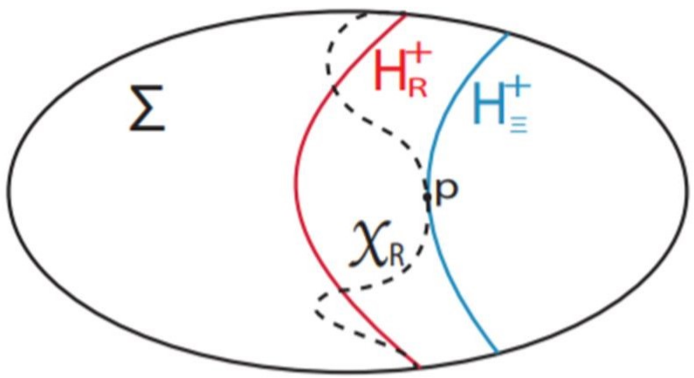


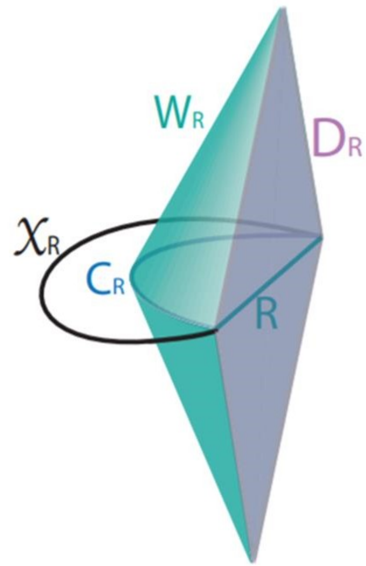
Theorem 4.1. *A quantum extremal surface $\mathcal{X}_R$ can never intersect W_R. The surface $\mathcal{X}_R$ is moreover generically spacelike separated from the causal surface C_R, but might be null separated from or coincide with C_R in non-generic spacetimes.*

Assume for contradiction that $C_R \cap \ \mathbf{Int}(\mathcal{X}_R) \neq \emptyset$.

$$\frac{\delta S_{\text{gen}}(N(\mathcal{X}_R))}{\delta n^a} k^a \geq \frac{\delta S_{\text{gen}}(H_\Xi^+)}{\delta n^a} k^a,$$

$$\frac{\delta S_{\text{gen}}(H_\Xi^+)}{\delta n^a} k^a \geq 0, \qquad \frac{\delta S_{\text{gen}}(H_\Xi^+)}{\delta n^a} k^a \leq 0,$$





Theorem 4.2. *The quantum apparent horizon always lies inside the horizon.*