

* 贝肯斯坦提出有界物质系统的熵 - 能以的普适熵界, 称为贝肯斯坦熵界

$$\frac{S}{E} \leq 2\pi R$$

S , E , R 为空间尺度

非黑洞熵界.

* 't Hooft 熵界: - 一个视界面积为 A 的区域所含的熵

$$\text{满足 } S \leq A/4$$

BH 熵是这种区域.

* Bousso 熵界 $S_L \leq |A_B - A_{B'}|/4$

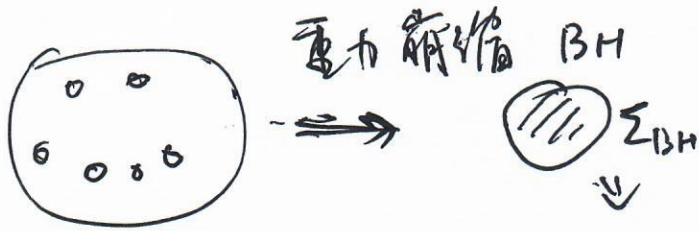
L 是起于类空横截面 B , 止于类空横截面 B' 的类光超曲面, 其膨胀熵 θ 处于非正或非止.

S_L 为穿过 L 流向 B 的类空或 B' 的类空的熵通量.

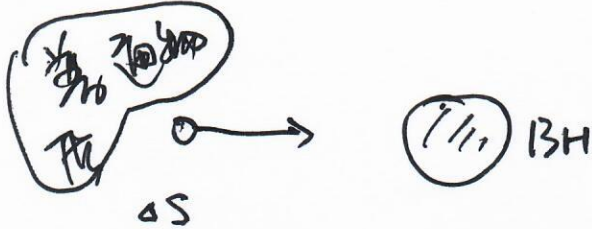
A_B 和 $A_{B'}$ 分别是 B 和 B' 的面积.

Holography.

$$S_{BH} = \frac{A(\Sigma_{BH})}{4G_N}$$



黑洞包括的信息是 S_{BH}



$$S_{BH} + \Delta S \leq \frac{A}{4}$$

ΔS 由贝肯斯坦提出的有界物体的熵一般的普朗克

界. $\frac{\Delta S}{\Delta E} \leq 2\pi R$ R 为该系统的半径.

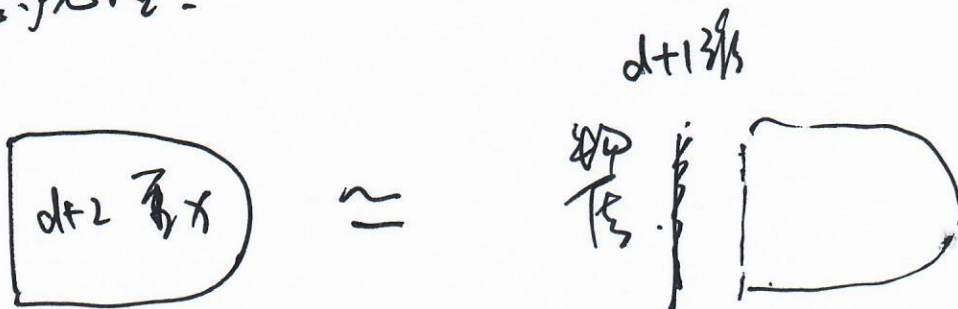
由 't Hooft bound. $S_{BH} + \delta S \leq \frac{A}{4}$

or Bousso 熵界.

\Rightarrow [BH 的熵/度]

\Rightarrow 重力理论的自由度 $\sim \frac{A}{4G}$.

全息原理:



QFT. spin +8 mod

$d+2$ 维的 gravity theory \approx $(d+1)$ 维的 $(\frac{d}{4})$ 维的

AdS 1+1 (Anti DeSitter Space)

$$\mathbb{R}^{2, d+1} : ds^2 = -(dx_0)^2 - (dx_{d+2})^2 + (dx_1)^2 + \dots + (dx_{d+1})^2$$

(x^0, \dots, x^{d+2}) 内部坐标

$$x_0^2 + x_{d+2}^2 = (x_1)^2 + \dots + (x_{d+1})^2 + R^2 \quad \text{AdS 半径}$$

Global 坐标

Poincaré 坐标

$$x_0 = R \cosh \rho \cos t = \frac{z}{2} \left(1 + \frac{R^2 + |\vec{x}|^2 - x_0^2}{z^2} \right)$$

$$x_{d+2} = R \cosh \rho \sin t = R \frac{x_0}{z}$$

$$x_i = R \sinh \rho \Omega_i = R \cdot \frac{x_i}{z} \quad i=1 \dots d$$

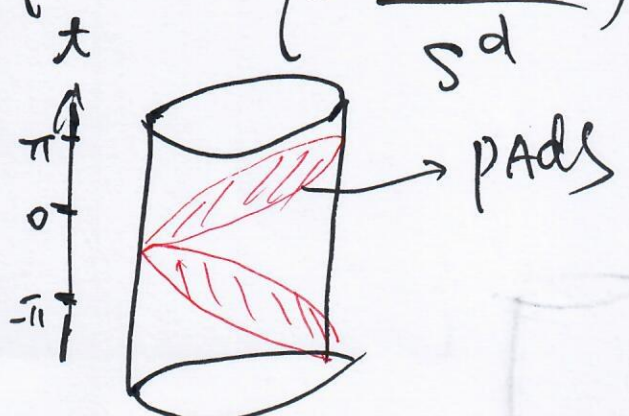
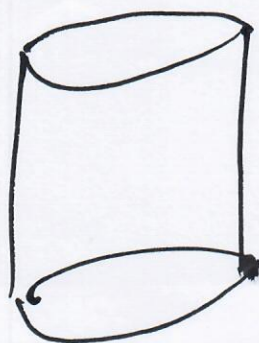
$$\sum_{i=1}^{d+1} (\Omega_i)^2 = 1$$

$$x_{d+1} = R \sinh \rho \Omega_{d+1} = \frac{z}{2} \left(1 - \frac{R^2 - |\vec{x}|^2 + x_0^2}{z^2} \right)$$

Global AdS

$$ds^2 = R^2 \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho \frac{(d\Omega_d)^2}{S^d} \right)$$

$$\mathcal{V}(g_{\text{AdS}_{d+2}}) = R \times S^d$$



AdS space

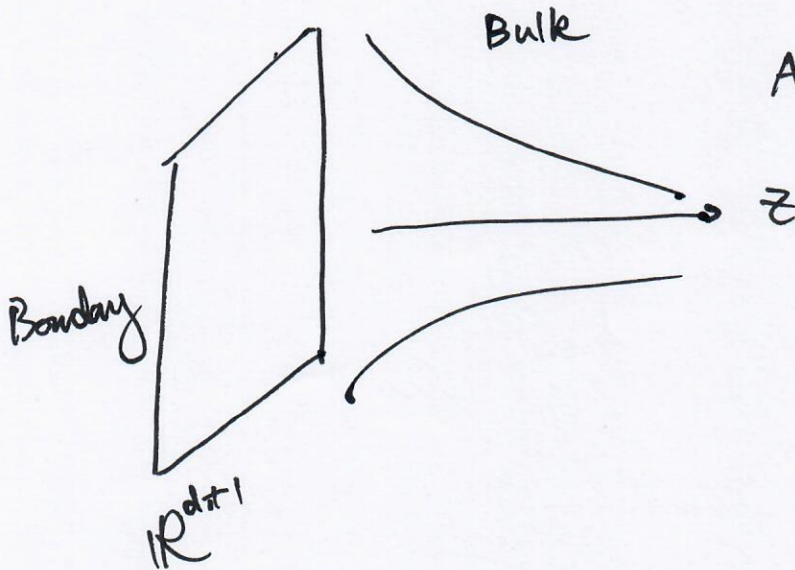
Maximally symmetric solution to the vacuum Einstein eq.
With a negative cosmological constant.

$$S = \frac{1}{16\pi G_N} \int dx \sqrt{-g} [R - 2\Lambda] \quad \Lambda = \frac{-(d+1)d}{2R^2}$$

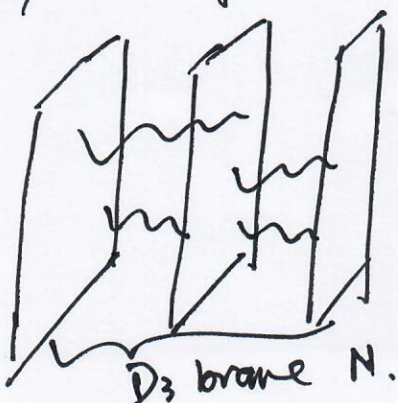
The metric of AdS_{d+2} (in Poincaré coordinate) is given

$$dS_{AdS_{d+2}}^2 = R^2 \left(\frac{dz^2 - dx_0^2 + \sum_{i=1}^d dx_i^2}{z^2} \right)$$

$$R_{\mu\nu\rho\sigma} = -R^2 (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho})$$



AdS/CFT from string theory



open string equivalent \Leftrightarrow



Gauge theory (CFI) = String Theory (Gravity) in AdS space.

Typically $SU(N)$ gauge theory in large N limit

$N=4$, 4d $SU(N)$ SYM \Leftrightarrow Type IIB $AdS_5 \times S^5$

D3-brane 解: $\alpha' = l_s^2$

$$ds^2 = \frac{1}{\sqrt{H(r)}} \sum_{\mu=0}^3 dx^\mu dx_\mu + \sqrt{H(r)} (dr^2 + r^2 d\Omega_5^2)$$

$N = \int_{S^5} *F_5$

$$H(r) = 1 + \frac{R^4}{r^4}$$

$$R^4 = 4\pi (\alpha')^2 N g_s$$

$\xrightarrow{r \rightarrow 0}$ 近视界极限

$$ds^2 \cong \underbrace{\frac{R^2}{r^2} (dx^\mu dx_\mu)}_{AdS_5} + \underbrace{\frac{R^2}{r^2} dr^2}_X + \underbrace{R^2 (d\Omega_5)^2}_{S^5}$$

$$g_s = g_{YM}^2$$

$$I = -\frac{1}{4g_{YM}^2} \text{Tr} (F^{\mu\nu} F_{\mu\nu}) + \dots$$

经典 引力 $\left\{ \begin{array}{l} \frac{R}{l_s} \gg 1 \\ \frac{R}{l_p} \gg 1 \end{array} \right. \rightarrow \lambda \equiv N g_{YM}^2 \gg 1 \text{ 强耦合}$

$$\frac{R}{l_p} \gg 1 \rightarrow N \gg 1, \quad l_p = g_s^{1/4} l_s$$

$$S_{\text{IIB}} = \frac{1}{4\kappa_B^2} \int \sqrt{|G|} e^{-2\Phi} (2R_G + 8\partial_\mu\Phi\partial^\mu\Phi - |H_3|^2)$$

$$- \frac{1}{4\kappa_B^2} \int \left[\sqrt{|G|} (|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|F_5|^2) + A_4^+ \wedge H_3 \wedge F_3 \right]$$

$$F_1 = dC \quad H_3 = dB \quad F_3 = dA_2 \quad F_5 = dA_4^+$$

$$\tilde{F}_3 = F_3 - CH_3, \quad \tilde{F}_5 = F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B \wedge F_3$$

对称性的自对偶条件 $*\tilde{F}_5 = \tilde{F}_3$

C 是零形式 RR-form Axion.

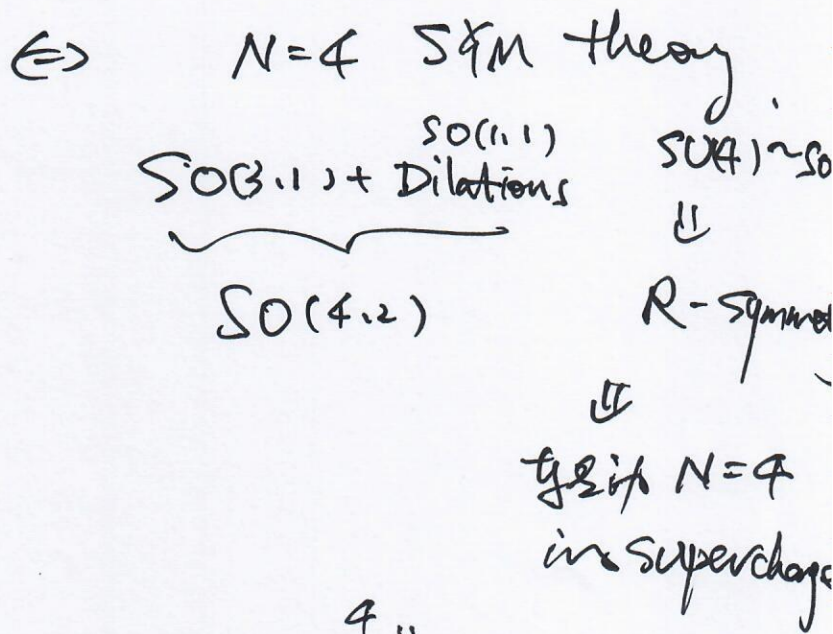
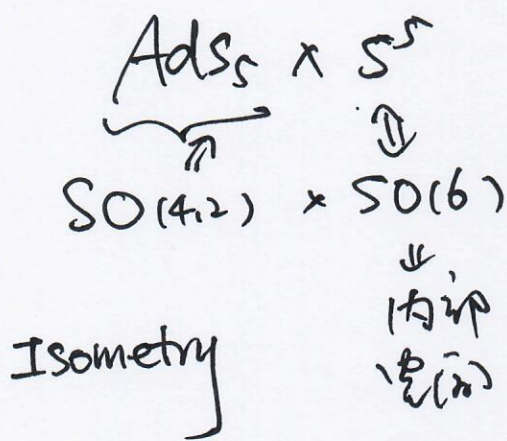
target space time. IIB SUPERGRA

N - D_3 -brane.

$$S = \frac{1}{(2\pi)^7 l_s^8} \int d^{10}x \sqrt{-g} (e^{-2\Phi} (R + 4|\partial\Phi|^2)$$

$$- \frac{2}{(8-p)!} F_{p+2}^2) \quad \text{where } p=3 \text{ ?}$$

AdS/CFT 对偶性 Matching



Fermionic sym.
 32 supercharge

 $2^{10/2}$

$4^{10} \cdot \frac{1}{2} \times 4 \times 2 = 32$
 $\underbrace{\hspace{10em}}_N$
 SCFT

SL(2, Z)

$\tau = \frac{i}{g_s} + \frac{C_0}{2\pi}$
 \Downarrow
 Dilaton $g_s = g_{YM}^2$
 $C_0 \Rightarrow$ axion.

$S: g_{YM}^2 \rightarrow \frac{16\pi^2}{g_{YM}^2}$

$T: \theta \rightarrow \theta + 2\pi$

\Downarrow
 CP 对偶性.

\Downarrow
 Montonen-olive Dual
 S-duality
 Geometric Langlands duality

$N=4$ SYM

Action is following

$$\begin{aligned} \mathcal{L} = \text{tr} \left\{ & -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta_2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right. \\ & - \sum_a i \bar{\lambda}^a \sigma^\mu D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i \\ & + \sum_{a,b,i} g C_i^{ab} \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] \\ & \left. + \frac{g^2}{2} \sum_{i,j} [X^i, X^j]^2 \right\} \end{aligned}$$

A_μ + 6 Scalars (Real) + 4 Fermion

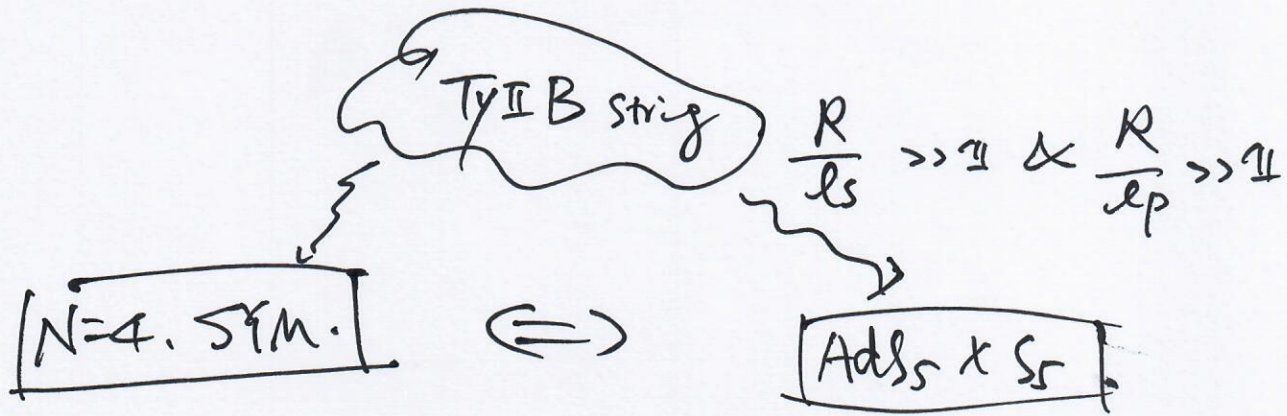
Symmetry of $S^5 \Leftrightarrow SO(6)$ R symmetry

adjoint representation of $SO(6)$

Further AdS_{d+2}/CFT_{d+1} Duality.

- $AdS_4/CFT_3 \Rightarrow$ ABJM theory CS + matter with SUSY
- AdS_3/CFT_2 , • JT + Matter \Leftrightarrow SYK model

d	CFT	bulk	String	weak
2	D-D5	$AdS_3 \times S^3 \times T^4$	c	c
3	ABJM	$AdS_4 \times S^7$	$N^{3/2}$	$\leftarrow N^2 (UV)$
4	$N=4$ SYM	$AdS_5 \times S^5 (IB)$	N^2	N^2
6	(2,0) Model	$AdS_7 \times S^4 (M)$	N^3	$\rightarrow N^2 (IR)$



GKPW 98'

$$\mathcal{Z}_{\text{CFT}}(\partial M) \stackrel{GKPW 98'}{=} \mathcal{Z}_{\text{Gravity}}(M)$$

$$= \langle e^{\int d^4x [\mathcal{L}_{\text{gauge}} + \phi^{(0)} \mathcal{O}(x)]} \rangle = \int \mathcal{D}g_{\mu\nu} \mathcal{D}\phi e^{-S(g, \phi)}$$

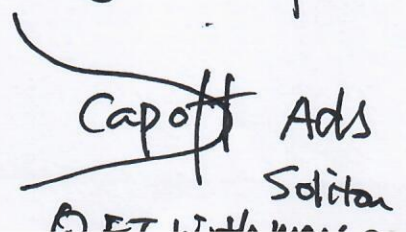
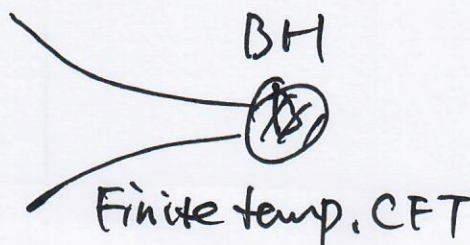
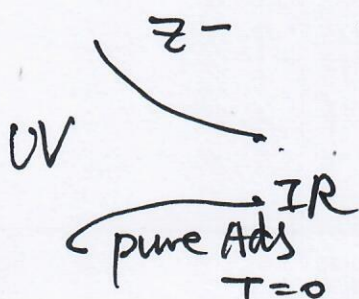
$$\approx e^{-S(g, \phi)} \Big|_{\text{on-shell}}$$

$$\phi(x, z) \Big|_{z \rightarrow 0} = \phi^{(0)} z^{\Delta_-} + \dots + \mathcal{O}(z^{\Delta_+})$$

$$\Delta = 2 \pm \sqrt{4 + m^2 R^2} \quad d=4$$

基本形式 AdS/CFT

AdS 空間 \Leftrightarrow generic QFT with UV fixed point



E.g. 我们不知道 # of degrees of freedom bounded
 by $N = \frac{A}{4G_N}$. For AdS_5 space.

$$G_N^{(5)} = \frac{G_N^{(10)}}{V_5} \sim \frac{g_5^2 (\alpha')^4}{R^5}$$

II B string theory in $AdS_5 \times S^5$.

$$16\pi G_N^{(10)} = (2\pi)^7 g_5^2 (\alpha')^4$$

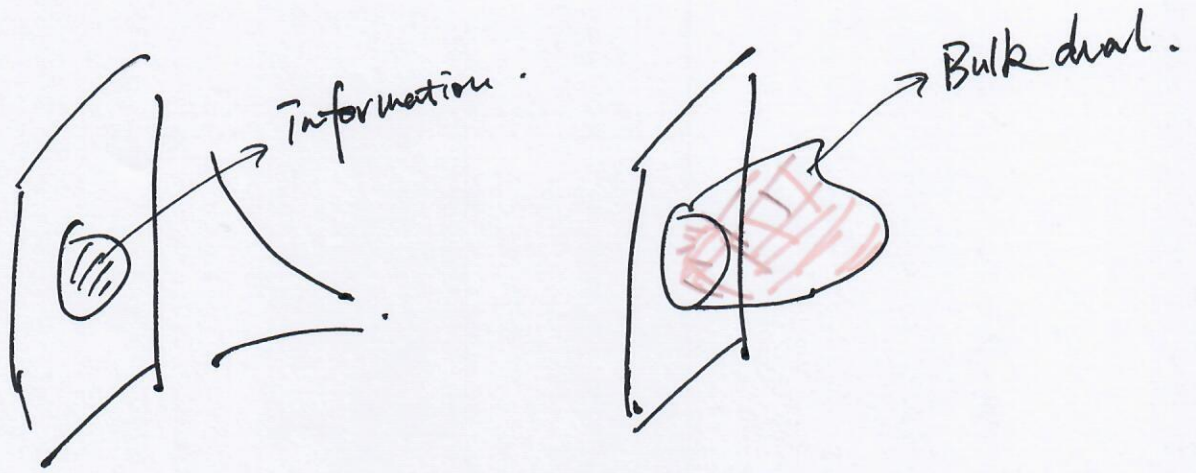
\Rightarrow 5D 牛顿常数.

又 \because 量子化的自由度 $N_1 = \frac{A}{4G_N} \sim A \cdot \frac{R^5}{g_5^2 (\alpha')^4}$

其中利用全息原理:

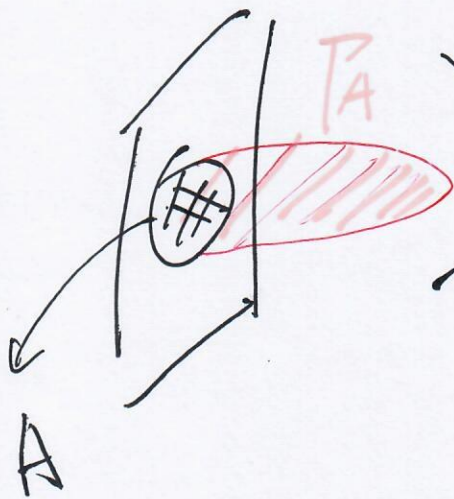
$$\begin{aligned} N_1 &\sim \frac{A \cdot R^5}{g_5^2 (\alpha')^4} = A \cdot \frac{R^8 \cdot R^{-3}}{g_{\text{YM}}^4 (\alpha')^4} = A \cdot \frac{R^{-3} (g_{\text{YM}}^2 N)^2}{g_{\text{YM}}^4} \\ &= \frac{A}{R^3} N^2 \\ &= \frac{L^3}{g^3} N^2 \end{aligned}$$

- T 的 问题是 ?



RT 2006. formula.

static 静态 Ads 时空



$$S_A = \text{Min}_{P_A} \left[\frac{A(P_A)}{4G_N} \right]$$

Min. 极小

\vec{P}_A 是 co-dim = 2 的曲面.

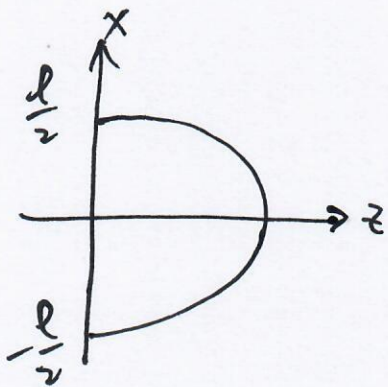
$\partial P_A = \partial A$ $\vec{P}_A \sim A$ 中间不包含 Singular Region

$$ds^2 = R^2 \left(\frac{dz^2 + dx^\mu dx_\mu}{z^2} \right) = R^2 \left(\frac{dz^2 - dt^2 + dX_i^2}{z^2} \right)$$

$$A(P_A) = R^d \int_{\epsilon}^{z_{\text{cut off}}} \frac{dz}{z^{2d/2}} \int_{\partial A} d^{d-1} x \Rightarrow S_A = \frac{R^d}{4G_N} \frac{A(\partial A)}{(d-1) \epsilon^{d-1}}$$

e.g. For 2D CFT

$$ds^2 = \frac{R^2}{z^2} (-dt^2 + dx^2 + dz^2)$$



$$ds^2 = \frac{R^2}{z^2} (dx^2 + dz^2)$$

其中 $\sqrt{x(z)}$ 或 $z(x) = z$

$$\text{induced: } ds^2 = \frac{R^2}{z^2} \left(1 + \left(\frac{dz}{dx} \right)^2 \right) (dx)^2$$

$$S = R \int_{-l/2}^{l/2} \frac{1}{z} \sqrt{1 + (z')^2} dx$$

$$\text{EOM: } z z'' + (z')^2 + 1 = 0$$

$$x = \pm \frac{l}{2}, \quad z = 0. \quad \Rightarrow \text{geodesic line}$$

$$z^2 + x^2 = \frac{l^2}{4}$$

Circle

$$ds^2 \Big|_{\mathbb{P}_A} = R^2 \times \frac{l^2}{z^2 (l^2 - 4z^2)} dz^2$$

$$A(\mathbb{P}_A) = 2R \int_{\epsilon}^{l/2} \frac{dz}{z} \times \frac{l}{\sqrt{l^2 - 4z^2}} = 2R \log \frac{l}{\epsilon}$$

$\frac{C}{3} \log \frac{l}{\epsilon}$
 $C = \frac{3R}{2\pi}$

Higher-dim Strip.



Poincaré coordinate

$$ds^2 = \frac{R^2}{z^2} (dz^2 - dt^2 + dx_i^2)$$

Strip $x_1 \in [-\frac{l}{2}, \frac{l}{2}]$

$x_{2,3}, \dots, dt = (-\infty, +\infty)$

$$ds^2 = \frac{R^2}{z^2} ((1+(z')^2) dx_1^2 + dx_2^2 + \dots)$$

~~$A = R^d L^{d-1}$~~

$$A = R^d L^{d-1} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{z^d} \sqrt{1+(z')^2} dx_1$$

EOM: $z z'' + d(z')^2 + d = 0$

$$\frac{dz}{dx} = \sqrt{\frac{c - z^{2d}}{z^{2d}}}$$

c is constant.

$$\left. \frac{dz}{dx} \right|_{z=z_*} = 0$$

$$\Rightarrow \frac{dz}{dx} = \sqrt{\frac{z_*^{2d} - z^{2d}}{z^{2d}}}$$

$$\Rightarrow \frac{l}{2} = \int_0^{z_*} dz \frac{z^d}{\sqrt{z_*^{2d} - z^{2d}}} = \frac{\sqrt{\pi} \Gamma(\frac{d+1}{2})}{\Gamma(\frac{1}{2d})} z_*$$

$$A = R^d L^{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \frac{z_*^d}{z^{2d}}$$

$$= 2R^d L^{d-1} \int_{\frac{L}{2}}^{z_*} \frac{z_*^d}{z^d} \frac{dz}{(z_*^{2d} - z^{2d})^{1/2}}$$

$$= 2R^d L^{d-1} \int_{\frac{L}{2}}^1 z_*^{1-d} dx x^{1-d-1} (1-x^{2d})^{\frac{1}{2}-1}$$

$$= 2R^d L^{d-1} \frac{B(\frac{1-d}{2d}, \frac{1}{2})}{2d} z_*^{1-d} + \frac{2R^d}{d-1} \left(\frac{L}{a}\right)^{d-1}$$

$$S_A = \frac{1}{4\epsilon_N^{(d+2)}} \left[\frac{2^d \pi^{d/2} R^d}{1-d} \left(\frac{\Gamma(\frac{d+1}{2d})}{\Gamma(\frac{1}{2d})} \right)^d \left(\frac{L}{a}\right)^{d-1} \right.$$

$$\left. + \frac{2R^d}{d-1} \left(\frac{L}{a}\right)^{d-1} \right]$$

↓
 发报! ⇒ Area law

EE for sphere

$$A = R^d \text{Vol}(S^{d-1}) \cdot \int_0^l dr \frac{r^{d-1}}{z^d} \sqrt{1 + \left(\frac{dz}{dr}\right)^2}$$

$$\Rightarrow \text{EOM: } r z z'' + (d-1) z (z')^2 + (d-1) z z' + dr (z')^2 + dr = 0$$

minimal surface $\Rightarrow r^2 + z^2 = l^2$

$$\text{Area} = \text{Vol}(S^{d-1}) R^d \int_{l/2}^1 dy \frac{(1-y^2)^{(d-2)/2}}{y^d}$$

$$= \frac{2\pi^{d/2} R^d}{\Gamma(d/2)} \left[\frac{1}{d-1} \left(\frac{l}{a}\right)^{d-1} - \frac{d-2}{2(d-3)} \left(\frac{l}{a}\right)^{d-3} + \dots \right]$$

$$S_A = \frac{2\pi^{d/2} R^d}{4G_N^{(d+2)} \Gamma(d/2)} \int_{l/2}^1 dy \frac{(1-y^2)^{\frac{d-2}{2}}}{y^d}$$

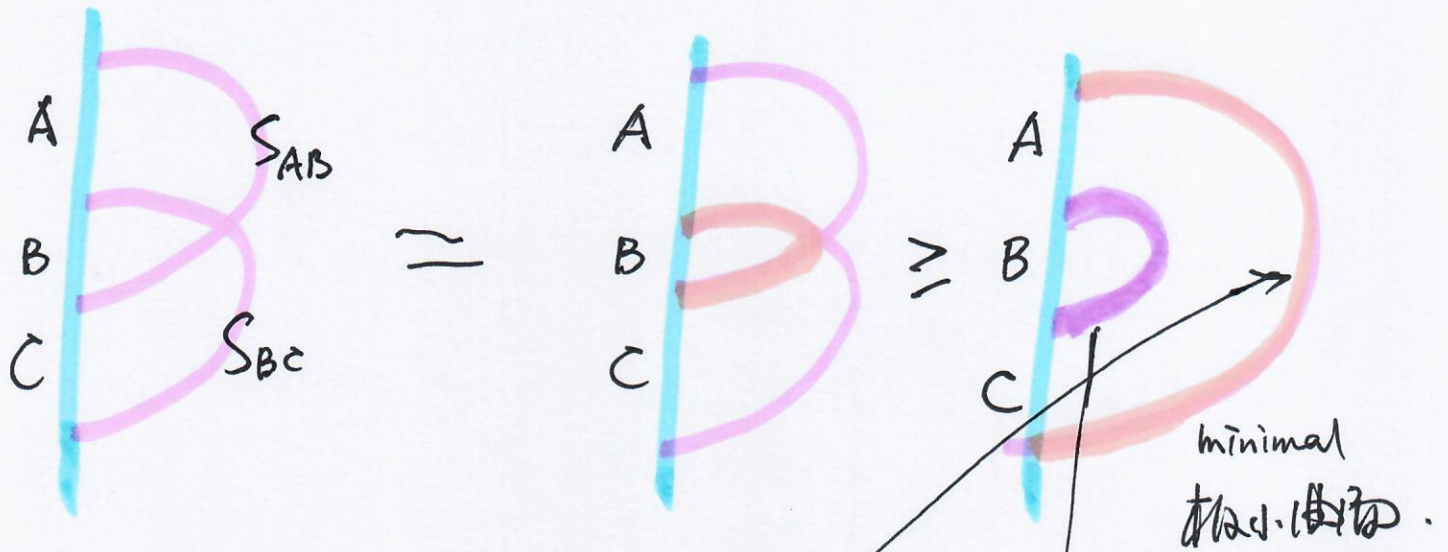
$$= \left\{ \begin{array}{l} P_{d+1} \left(\frac{l}{a}\right)^{d-1} + P_2 \left(\frac{l}{a}\right)^{d-3} + \dots \text{Central charge } d \text{ even} \\ P_1 \left(\frac{l}{a}\right)^{d-1} + P_2 \left(\frac{l}{a}\right)^{d-3} + \dots \text{ } (-1)^{\frac{d-1}{2}} F \text{ } d \text{ odd} \end{array} \right.$$

= proportional to $\frac{1}{\epsilon}$

Strong subadditivity

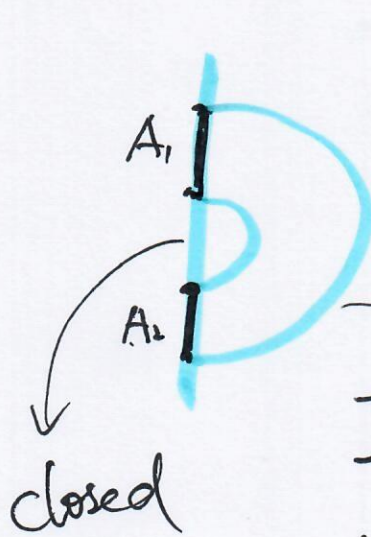
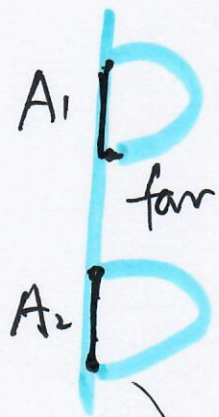
Strong subadditivity

Headridge - Tadashi 2009



$$S_{AB} + S_{BC} \geq S_{ABC} + S_B$$

phase transition. $A = A_1 \cup A_2$ (disconnected sum)

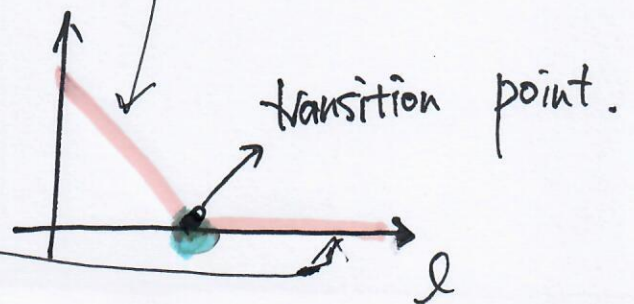


Consistent with

[Calabrese - Cardy - Tonni 2007]

$$I(A_1 : A_2) = S_{A_1} + S_{A_2} - S_{A_1 \cup A_2}$$

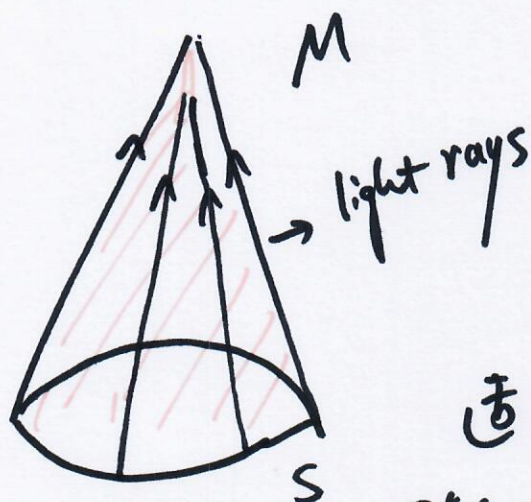
l is distance of A_1, A_2



light sheet

类光测地线族构成的子流形

具有非止的膨胀 (expansion)



HRT formula.

适用于一般含时间依赖的
渐近 AdS 时空。

Covariant HEE

$$S_A = \underset{P_A^{\text{ext}}}{\text{Min}} \underset{P_A}{\text{Ext}} \left[\frac{A(P_A)}{4G_N} \right]$$

Extremal surface

中选择面积最小的。

Lorentzian AdS 中的极值
曲面 (Extremal surface)

可应用非平行输运过程。



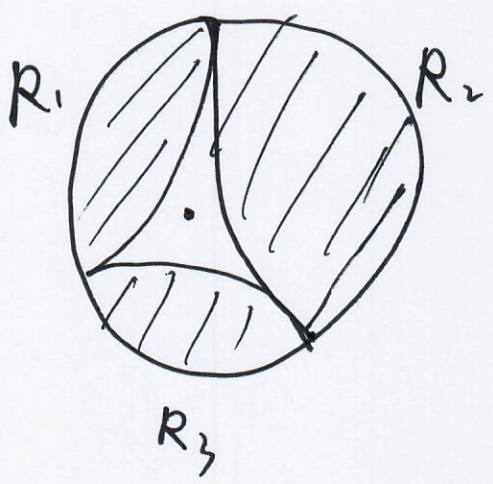
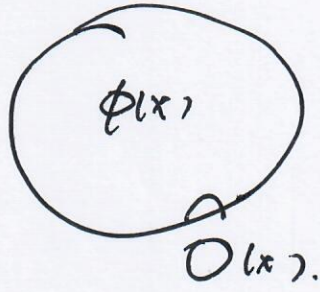
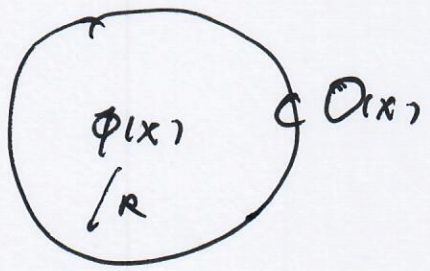
Comment: 由解志 → 非解志。

极小曲面 → 外曲率
= 0
的超曲面。

puzzle of the bulk locality

Space like separated $\Rightarrow [\phi(x), \mathcal{O}(x)] = 0$

\Downarrow Schur's lemma
 $\phi(x) = I$



$$[\phi(x), \mathcal{O}(x)] = 0$$

code subspace

$$S_{\dots} = \frac{-\text{Tr}[\partial_n \hat{\rho}_n] |_{n=1}}{\text{Tr}[\hat{\rho}_1]}$$

为3it的 S_{\dots} 利用 Lagrangian formalism.

$$\mathcal{L}(\hat{g}_n, h, \varphi)$$

metric 扰动 \rightarrow 其他物扰动

$$S_{\dots} = \underbrace{\langle \int d\tau \partial_n \mathcal{L} \rangle}_{\text{对反冲的 path-Int}} = \int d\tau \langle E_{\mu\nu}(\hat{g} + h, \varphi) \partial_n \hat{g}^{\mu\nu} + d\text{④}(\hat{g}, h, \varphi, \partial_n \hat{g}) - \int d\tau d\text{④}(\hat{g}, \partial_n g) \rangle$$

$E_{\mu\nu}$ 是反冲的 $\neq 0$ Due to Quantum fluctuation.

④ 是熵. Associated with Wald-like entropy.

$$\text{第1步. } \int d\tau \langle E_{\mu\nu} \rangle \partial_n \hat{g}^{\mu\nu} = -\frac{1}{2} \int d\tau \langle T_{\mu\nu} \rangle \partial_n \hat{g}^{\mu\nu}$$

$$\hat{g} \rightarrow \hat{g} + \bar{h}$$

$$\Rightarrow E_{\mu\nu}(\hat{g} + \bar{h}) = -\langle E_{\mu\nu} \rangle = \frac{1}{2} \langle T_{\mu\nu} \rangle$$

路径的配分函数

$$Z_{g,n} = \text{Tr} \left[P e^{-\int_0^{2\pi n} dt \cdot H_{b,n}(t)} \right] = \text{Tr} [\hat{\rho}_n^n]$$

$$\hat{\rho}_n = P e^{-\int_0^{2\pi} H_{b,n}(t) dt} \quad H_{b,n}(t) = H_{b,n}(t+2\pi)$$

$H_{b,n}$ 是时间依赖的 Hamiltonian.

Bulk 的 density matrix $\hat{\rho}_n$

$\hat{\rho}_n$ 对应的 Bulk geometry 是 $\hat{g}_n \cong g_n / \mathbb{Z}_n$.

忽略 UV divergencies.

引入 UV regulator 是 Local. Covariant.

$$\text{熵 } S_g = -\partial_n (\log Z_{g,n} - n \log Z_{g,1})_{n=1}$$

$$= -\partial_n (\log \text{Tr} [\hat{\rho}_n^n] - n \log \text{Tr} [\hat{\rho}_1])_{n=1}$$

$$= S_{\text{Bulk-ent}} + S_{\dots}$$

$$\downarrow$$
$$= -\partial_n (\log \text{Tr} [\rho_1^n] - n \log \text{Tr} [\rho_1])_{n=1}$$

$$\int dt \int E(g^{\hat{a}} + \hbar) \partial_n g^{\hat{a}} = \partial_n I_n(g^{\hat{a}} + \hbar) |_{r=1} - \int dt d\Omega (g^{\hat{a}} + \hbar) \partial_n g^{\hat{a}}$$

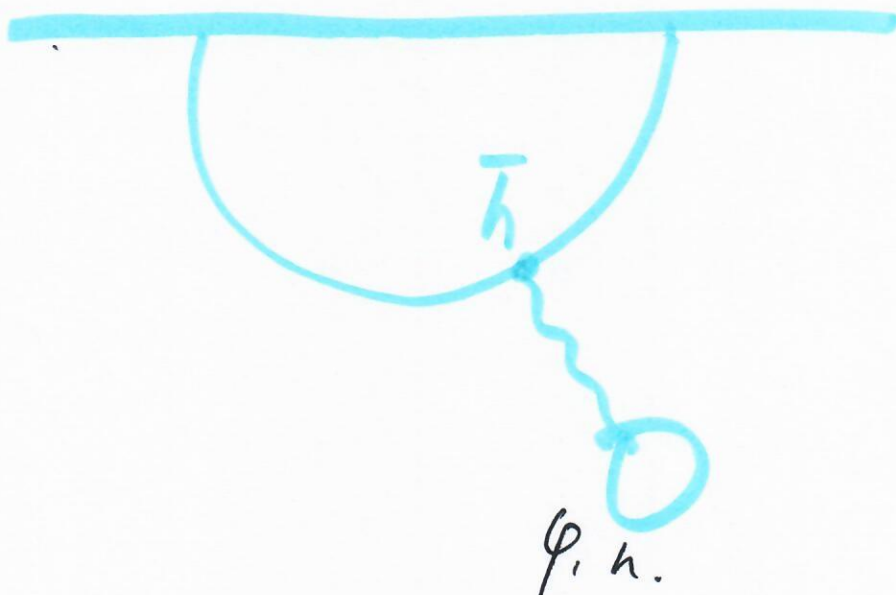
$$I_n(g^{\hat{a}} + \hbar) = I_n(g^{\hat{a}}) \text{ up to 1st order,}$$

\Downarrow
 对应着移的项 $\frac{\delta A}{4G_N}$

若是有高阶的项则对应着.

SSW-like..

A



还需要引入 $\frac{1}{\epsilon^2} R$ in counter terms.

$$\frac{\text{Area}}{\epsilon^{D-2}}$$

is make Bulk quantum gravity finite