Plan of Lectures

- I **Overview**
  - Lecture duration ~ 1 hr
- II **What are GWs?**
  - Lecture duration ~ 2 hr
- III **Gravity Tests with GWs**
  - Lecture duration ~ 1.5 hr

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That would be one of the most fascinating things man could do, because it would tell you very much how the universe started.

— Rainer Weiss

北京大學
PEKING UNIVERSITY

Frontiers of GWs (II): What are GWs?

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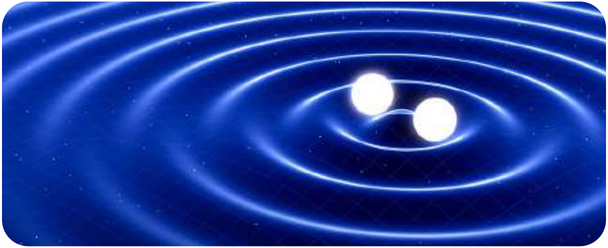
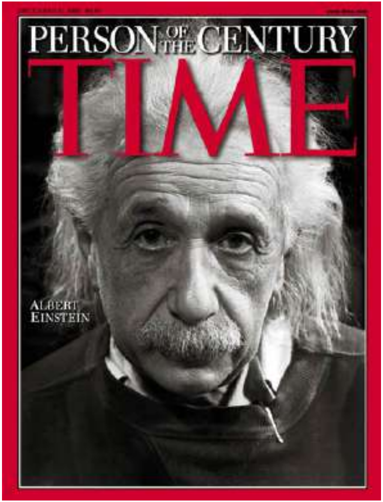
KITS Summer School · 江苏溧阳

References

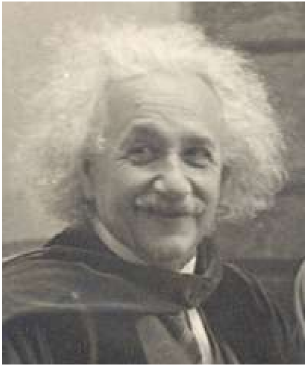
- M. Maggiore, *Gravitational Waves* (Volume 1: Theory and Experiments), Oxford University Press (2008)
- M. Maggiore, *Gravitational Waves* (Volume 2: Astrophysics and Cosmology), Oxford University Press (2018)
- A. Buonanno, Les Houches Lecture Notes (2006) [arXiv:0709.4682]



Person of the Century



General Relativity



$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

"Matter tells spacetime how to curve, and spacetime tells matter how to move."

Einstein Field Equations

Einstein Field Equations in a Nutshell

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where

$$\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} \\
R_{\mu\nu} &= g^{\rho\sigma} R_{\rho\mu\sigma\nu} \\
R^{\nu}{}_{\mu\rho\sigma} &= \Gamma^{\nu}{}_{\mu\sigma,\rho} - \Gamma^{\nu}{}_{\mu\rho,\sigma} + \Gamma^{\nu}{}_{\lambda\rho}\Gamma^{\lambda}{}_{\mu\sigma} - \Gamma^{\nu}{}_{\lambda\sigma}\Gamma^{\lambda}{}_{\mu\rho} \\
\Gamma^{\mu}{}_{\nu\rho} &= \frac{1}{2} g^{\mu\lambda} \left(g_{\lambda\nu,\rho} + g_{\lambda\rho,\nu} - g_{\nu\rho,\lambda} \right)
\end{aligned}$$

Linearized Gravity

Perturbation of $g_{\mu\nu}$

- In order to study GWs, we assume there exists a coordinate system where the spacetime of interests has[1]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \left|h_{\mu\nu}\right| \ll 1$$

- Consider a Lorentz transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, we have

$$g_{\mu\nu} \to g'_{\mu\nu}\left(x'\right) = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu g_{\rho\sigma} = \eta_{\mu\nu} + \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu h_{\rho\sigma}(x) = \eta_{\mu\nu} + h'_{\mu\nu}\left(x'\right)$$

  where we have used $\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} = \eta_{\mu\nu}$

- Therefore, $h_{\mu\nu}$ can be viewed as a tensor field in a flat spacetime

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[1] More later ;-)

Perturbation of $g_{\mu\nu}$

- Now consider a coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \quad \left|\partial_\mu \xi_\nu\right| \leq \left|h_{\mu\nu}\right|$$

- The metric becomes

$$g_{\mu\nu}(x) \to g'_{\mu\nu}\left(x'\right) = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

- Keeping leading-order terms,

$$g'_{\mu\nu} = \eta_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + h_{\mu\nu} + \mathcal{O}\left(\xi^2\right)$$

- Therefore, $h_{\mu\nu}$ satisfies

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}\,, \quad \left|h'_{\mu\nu}\right| \ll 1$$

Perturbation of $g_{\mu\nu}$

- Keeping the leading-order terms of $h_{\mu\nu}$, we have[2]

$$\begin{aligned}
\Gamma^{\nu}{}_{\mu\rho} &= \frac{1}{2}\eta^{\nu\lambda}\left(\partial_\rho h_{\lambda\mu} + \partial_\mu h_{\lambda\rho} - \partial_\lambda h_{\mu\rho}\right) \\
R^{\nu}{}_{\mu\rho\sigma} &= \partial_\rho \Gamma^{\nu}{}_{\mu\sigma} - \partial_\sigma \Gamma^{\nu}{}_{\mu\rho} + \mathcal{O}\left(h^2\right) \\
R_{\mu\nu\rho\sigma} &= \frac{1}{2}\left(\partial_{\rho\nu} h_{\mu\sigma} + \partial_{\sigma\mu} h_{\nu\rho} - \partial_{\rho\mu} h_{\nu\sigma} - \partial_{\sigma\nu} h_{\mu\rho}\right)
\end{aligned}$$

- A direct calculation shows that, under the change of $h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$, the Rieman tensor does not change

[2] Homework ;-)

Equation of GWs

- Define a trace-reverse tensor,

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h$$

which satisfies $h = \eta_{\alpha\beta} h^{\alpha\beta}$ and $\bar{h} = -h$

- With a linearized metric, the Einstein field equations become

$$\Box \bar{h}_{\nu\sigma} + \eta_{\nu\sigma}\partial^\rho \partial^\lambda \bar{h}_{\rho\lambda} - \partial^\rho \partial_\nu \bar{h}_{\rho\sigma} - \partial^\rho \partial_\sigma \bar{h}_{\rho\nu} + \mathcal{O}\left(h^2\right) = -\frac{16\pi G}{c^4} T_{\nu\sigma}$$

Equation of GWs

- Introduce Lorenz gauge (a.k.a. harmonic gauge, De Donder gauge)

$$\partial_\nu \bar{h}^{\mu\nu} = 0$$

- We finally obtain a wave equation

$$\Box \bar{h}_{\nu\sigma} = -\frac{16\pi G}{c^4} T_{\nu\sigma}$$

Equation of GWs

- If $\bar{h}^{\mu\nu}$ does not satisfy Lorenz gauge, namely

$$\partial_\mu \bar{h}^{\mu\nu} = q^\nu \neq 0$$

- We can always perform a coordinate transformation, s.t.

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\left(\partial_\rho \xi^\rho\right)$$

- as long as $\Box \xi_\nu = q_\nu$, we can obtain $\partial_\mu \bar{h}'^{\mu\nu} = 0$ (i.e., Lorenz gauge)
- **Lorenz gauge** reduces the d.o.f.s of $h_{\mu\nu}$ from **10** to **6**

Transverse Traceless Gauge

- In vacuum, $T_{\mu\nu} = 0$, therefore

$$\Box \bar{h}_{\mu\nu} = 0$$

  thus, GWs propagate with the speed of light

- On top of the Lorenz gauge, consider $x'^{\mu} = x^{\mu} + \xi^{\mu}$
  - as long as $\Box \xi_{\mu} = 0$, the Lorenz gauge is preserved
  - Now $\bar{h}_{\mu\nu}$ becomes

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \xi_{\mu\nu}$$

where $\xi_{\mu\nu} = \eta_{\mu\nu}\partial_\rho \xi^\rho - \xi_{\mu,\nu} - \xi_{\nu,\mu}$, satisfying $\Box \xi_{\mu\nu} = 0$

Transverse Traceless Gauge

- With it, d.o.f.s of $h_{\mu\nu}$ are reduced from **6** to **2**; specifically
  - choose ξ^0, s.t. $h = 0$ (now, $\bar{h}_{\mu\nu} = h_{\mu\nu}$)
  - choose ξ^i, s.t. $h^{i0} = 0$
- As for now, we know from the $\mu = 0$ component of the Lorenz gauge $\partial_\nu \bar{h}^{\mu\nu} = \partial_\nu h^{\mu\nu} = 0$ that $\partial_0 h^{00} = 0$ (we take $h^{00} = 0$)
- Overall, we call the following transverse-traceless gauge

$$h^{00} = h^{ii} = h^{0i} = 0, \quad \partial_i h^{ij} = 0$$

We denote GWs in TT gauge as h_{ij}^{TT}

h_{ij}^{TT}

- For a plane wave, $\partial_i h^{ij} = 0$ means $\hat{n}^i h_{ij}^{\mathrm{TT}} = 0$, where $\hat{\boldsymbol{n}} = \boldsymbol{k}/k$ is the propagating direction
- Without losing generality, we consider GWs propagating along z-axis, and we have

$$h_{ij}^{\mathrm{TT}}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\left[\omega\left(t - \frac{z}{c}\right)\right]$$

where h_+ and $h_\times$ are two independent **polarizations**

h_{ij}^{TT}

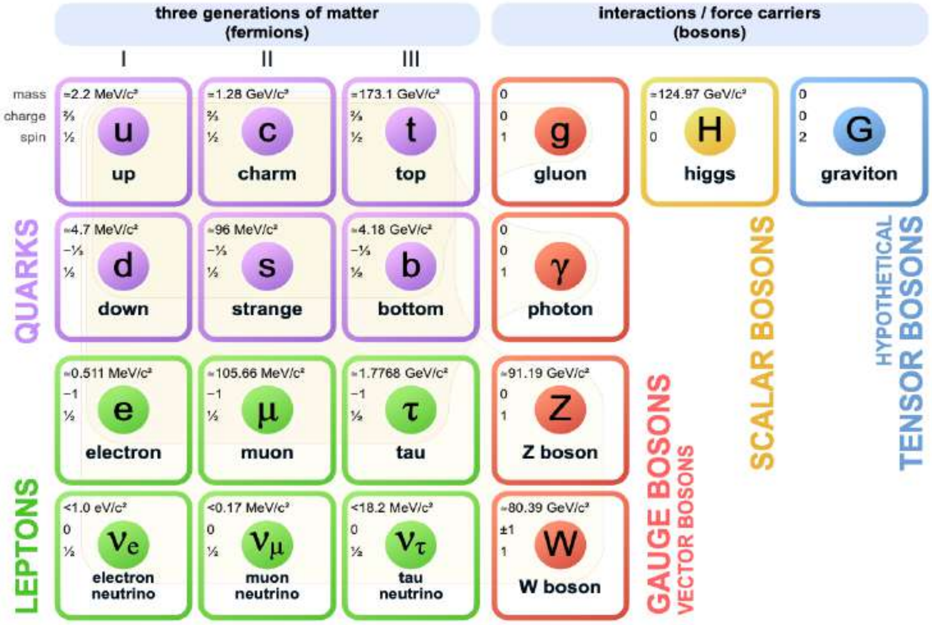
- If we rotate about z axis by an angle ψ,

$$h_\times \pm i h_+ \rightarrow e^{\mp 2i\psi} \left(h_\times \pm i h_+ \right)$$

  Therefore, gravitons are **spin-2** particles

- **What are gravitons?**

Standard Model of Elementary Particles and Gravity



Projection Operators

- For a given direction $\hat{\boldsymbol{n}}$, introduce

$$P_{ij}(\hat{\boldsymbol{n}}) = \delta_{ij} - \hat{n}_i \hat{n}_j$$
$$\Lambda_{ijkl}(\hat{\boldsymbol{n}}) = P_{ik} P_{jl} - \frac{1}{2} P_{ij} P_{kl}$$

- If h_{kl} describes GWs in **Lorenz gauge** (not necessarily **TT guage**), then

$$h_{ij} \equiv \Lambda_{ijkl} h_{kl}$$

satisfies TT gauge

GW Detections

- Now we consider a local free fall (FF) coordinate (note: not a TT gauge!)
  - In FF coordinate, we have $g_{\mu\nu}(P) = \eta_{\mu\nu}$ and $\Gamma^{\rho}{}_{\mu\nu}(P) = 0$
  - LIGO/Virgo/KAGRA are obviously not in a FF state
  - However , it is a good approximation for some frequency bands (e.g. $\sim 100\,\mathrm{Hz}$)
- Without proof,[3] we denote that for two nearby particles,

$$\frac{\mathrm{d}^2 \xi^j}{\mathrm{d}t^2} = \frac{1}{2} \ddot{h}^{\mathrm{TT}}_{jk} \xi^k$$

[3] It can be obtained from geodesic equation; see Sec. 3.3 in arXiv:0709.4682

GW Detections

- Consider particles on a ring whose norm is in z-direction
- With a "+" mode GW,

$$h_{ij}^{\mathrm{TT}} = h_+ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sin \omega t$$

- Relative to the center, particles' position becomes

$$\xi_i = [x_0 + \delta x(t), y_0 + \delta y(t)]$$

- According to the equation on the previous slide, we obtain[4]

$$\delta x(t) = \frac{h_+}{2} x_0 \sin \omega t$$
$$\delta y(t) = -\frac{h_+}{2} y_0 \sin \omega t$$

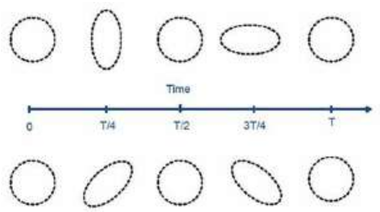
[4] Notice that, amazingly, now it is a Newtonian-like force!

GW Detections

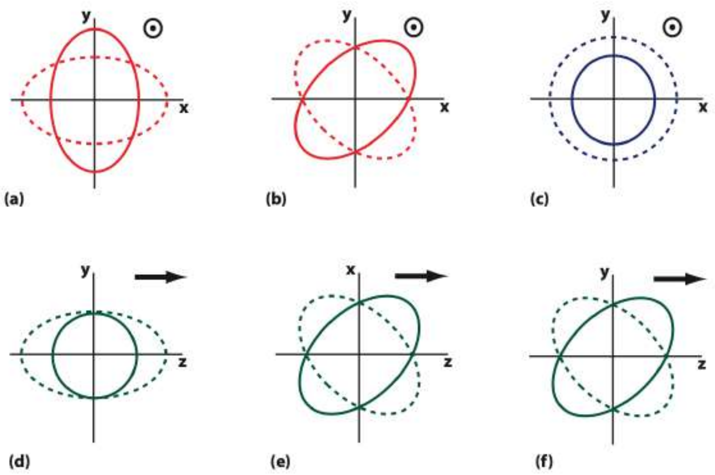
- Similarly, a "$\times$" mode GW gives

$$\delta x(t) = \frac{h_\times}{2} y_0 \sin \omega t$$
$$\delta y(t) = \frac{h_\times}{2} x_0 \sin \omega t$$

- Therefore, we have the positions of particles as a function of time,



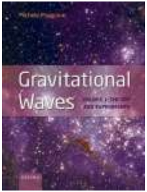
GW Polarizations in Alternative Gravity

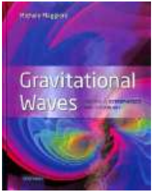


Eardley et al. 1973; Will 2014

GW Generation

- As GR is highly nonlinear, it is impossible to obtain analytic solutions in a generic setting
- Here we only present some simple results
- For more details, see
  - Buonanno's Les Houches Lecture arXiv:0709.4682
  - Michele Maggiore's books Gravitational Waves (Vol I & Vol II)





GW Generation

- Under Lorenz gauge, $\partial_\mu \bar{h}^{\mu\nu} = 0$
- Linearized Einstein equation becomes,

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- We make weak-field & slow-motion assumptions, and get

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ijkl}(\hat{\mathbf{n}})\ddot{M}^{kl}\left(t - \frac{r}{c}\right)$$

where

$$M^{ij} = \frac{1}{c^2}\int \mathrm{d}^3 x T^{00}(t, \mathbf{x}) x^i x^j$$

is mass quadrupole moment in Newtonian approximation

GW Generation

- Take $\hat{\mathbf{n}} = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$ and insert into the projection operator $\Lambda_{ijkl}(\hat{\mathbf{n}})$, then[5]

$$
\begin{aligned}
h_{+} = & \frac{G}{rc^4} \Big\{ \ddot{M}_{11} \left(\sin^2\varphi - \cos^2\theta \cos^2\varphi \right) \\
& + \ddot{M}_{22} \left(\cos^2\varphi - \cos^2\theta \sin^2\varphi \right) - \ddot{M}_{33} \sin^2\theta \\
& - \ddot{M}_{12} \sin 2\varphi \left(1 + \cos^2\theta \right) + \ddot{M}_{13} \cos\varphi \sin 2\theta + \ddot{M}_{23} \sin 2\theta \sin\varphi \Big\} \\
h_{\times} = & \frac{2G}{rc^4} \Big\{ \frac{1}{2} \left(\ddot{M}_{11} - \ddot{M}_{22} \right) \cos\theta \sin 2\varphi - \ddot{M}_{12} \cos\theta \cos 2\varphi \\
& - \ddot{M}_{13} \sin\theta \sin\varphi + \ddot{M}_{23} \cos\varphi \sin\theta \Big\}
\end{aligned}
$$

[5] Don't be afraid & take it homework ;-)

Binary Systems

- Consider a binary with masses m_1 and m_2
  - total $M = m_1 + m_2$, and reduced mass $\mu = m_1 m_2 / (m_1 + m_2)$
- Assume a circular orbit,

$$X(t) = R\cos\omega t, \quad Y(t) = R\sin\omega t, \quad Z(t) = 0$$

- Then the mass quadrupole moments are

$$\begin{aligned} M_{11} &= \frac{1}{2}\mu R^2 (1 + \cos 2\omega t) \\ M_{22} &= \frac{1}{2}\mu R^2 (1 - \cos 2\omega t) \\ M_{12} &= \frac{1}{2}\mu R^2 \sin 2\omega t \end{aligned}$$

Binary Systems

- Insert into expressions of h_+ & $h_\times$, we have

$$
\begin{aligned}
h_+(t) =& \frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2 \frac{\left(1+\cos^2\theta\right)}{2}\cos(2\omega t) \\
h_\times(t) =& \frac{1}{r}\frac{4G}{c^4}\mu R^2\omega^2 \cos\theta \sin(2\omega t)
\end{aligned}
$$

- These are the leading-order GW formuae that we frequently use

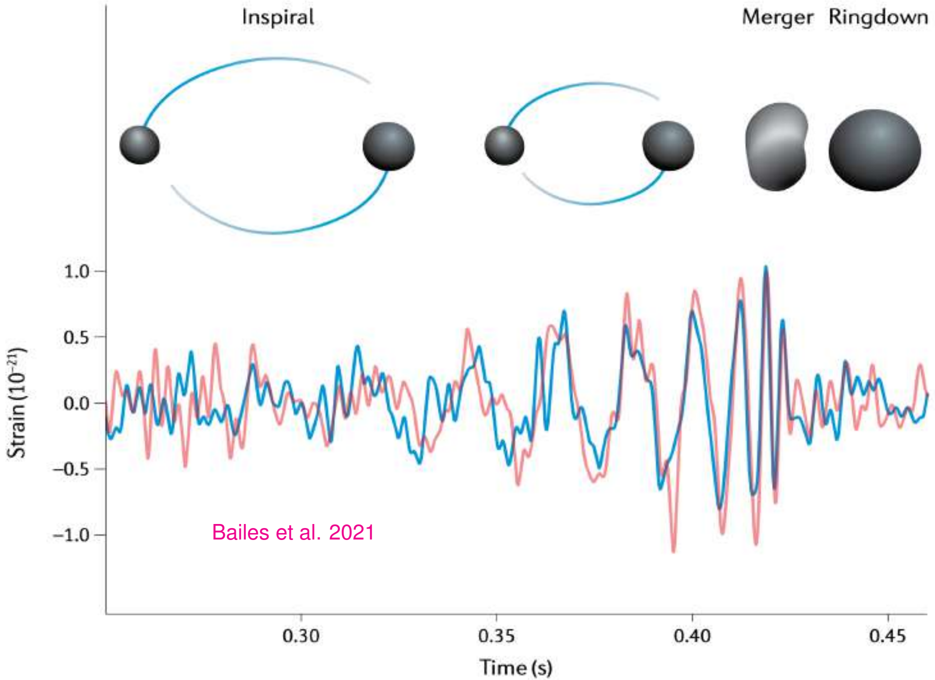
GW Radiation

- As GWs carry energy, the GW radiation reduces the binary's orbital energy $\Leftarrow$ energy balance equation
- The orbital size becomes **smaller**, and the orbital frequency becomes **larger**
- At leading order,[6] with $\nu \equiv \mu/M$ and $\mathcal{M} = \mu^{3/5}M$,

$$\frac{\dot{\omega}}{\omega^2} = \frac{96}{5}\nu\left(\frac{GM\omega}{c^3}\right)^{5/3}$$

$$\dot{f}_{\rm GW} = \frac{96}{5}\pi^{8/3}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3} f_{\rm GW}^{11/3}$$

[6] GW frequency is twice that of the orbit, $f_{\rm GW} = \omega/\pi$


Inspiral
Merger
Ringdown
1.0
0.5
0.0
−0.5
−1.0
Strain (10^−21)
0.30
0.35
0.40
0.45
Time (s)
Bailes et al. 2021

GWs on a curved spacetime

Separation of GWs from the Background

- On a **curved**, **dynamical** background metric

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x), \quad \left|h_{\mu\nu}\right| \ll 1$$

such that satisfying (short-wave expansion) $\lambda \ll L_B$ or $f \gg f_B$

- **Ricci tensor**: $\mathcal{O}(1)$, $\mathcal{O}(h)$, and $\mathcal{O}(h^2)$

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \ldots$$

  - $\bar{R}_{\mu\nu}$: only **low** frequency
  - $R^{(1)}_{\mu\nu}$: only **high** frequency
  - $R^{(2)}_{\mu\nu}$: **mixture of both**

Separation of GWs from the Background

- **Master equations**

$$\bar{R}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\mathrm{Low}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\mathrm{Low}}$$

$$R^{(1)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\mathrm{High}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\mathrm{High}}$$

- low-frequency equation $\Rightarrow$ energy-stress tensor of GWs
- high-frequency equation $\Rightarrow$ propagating equation of GWs

Low-frequency Equation

$$\bar{R}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\text{Low}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{Low}}$$

- If curvature is determined by GWs $\Rightarrow h \sim \frac{\lambda}{L_B}$
- If curvature is determined by matter fields $\Rightarrow h \ll \frac{\lambda}{L_B}$
- These two conclusions will be important for a later context

Low-frequency Equation

- Difficulties with **localized** **energy-stress tensor** in GR
- Learn from **renormalization group**

"coarse-grained" form of the Einstein equation

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}\left(\bar{T}_{\mu\nu} + t_{\mu\nu}\right)$$

where

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\right\rangle$$

Low-frequency Equation

- In Lorentz gauge and with $h = 0$, one has

$$t_{\mu\nu} = \frac{c^4}{32\pi G} \left\langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \right\rangle$$

- Explicit calculations show that,[7] $t_{\mu\nu}$ only depends on the physical modes h_{ij}^{TT} $\Rightarrow$ namely, gauge invariant
- Energy-momentum exchange between matters and GWs

$$\bar{D}^\mu \left(\bar{T}_{\mu\nu} + t_{\mu\nu} \right) = 0$$

[7] Use $x'^\mu = x^\mu + \xi^\mu$

Low-frequency Equation

- With **energy-stress tensor** for GWs, we can discuss many aspects of GWs, e.g.,

$$t^{00} = \frac{c^2}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle$$

- Similarly to the electromagnetism, we have

$$\frac{\mathrm{d}E}{\mathrm{d}A\mathrm{d}t} = +ct^{00}$$
$$\frac{\mathrm{d}P^k}{\mathrm{d}A\mathrm{d}t} = +t^{0k}$$

Low-frequency Equation

- We can have the energy radiation rate and the momentum taken away by GWs,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{c^3 r^2}{32\pi G} \int d\Omega \left\langle \dot{h}_{ij}^{\mathrm{TT}} \dot{h}_{ij}^{\mathrm{TT}} \right\rangle$$

$$\frac{dP^k}{dt} = -\frac{c^3}{32\pi G} r^2 \int d\Omega \left\langle \dot{h}_{ij}^{\mathrm{TT}} \partial^k h_{ij}^{\mathrm{TT}} \right\rangle$$

- as well as the energy spectrum

$$\frac{\mathrm{d}E}{\mathrm{d}f} = \frac{\pi c^3}{2G} f^2 r^2 \int d\Omega \left(\left| \tilde{h}_+(f) \right|^2 + \left| \tilde{h}_\times(f) \right|^2 \right)$$

and so on

High-frequency Equation

$$R^{(1)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\rm High} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\rm High}$$

- If $T^{\mu\nu} = 0$,[8] one has $R^{(1)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\rm High}$

$$R^{(1)}_{\mu\nu} \sim \partial^2 h \sim \frac{h}{\bar{\lambda}^2} \sim \frac{1}{\epsilon}$$

$$R^{(2)}_{\mu\nu} \sim \partial^2 h^2 \sim \frac{h^2}{\bar{\lambda}^2} \sim 1$$

- At leading order, $R^{(1)}_{\mu\nu} = 0 \Rightarrow \Box \bar{h}_{\mu\nu} = 0$ in Lorenz gauge

[8] Now, according to the low-frequency equation, one has $h \sim \bar{\lambda}/L_B$

High-frequency Equation

$$R^{(1)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^{\text{High}} + \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{High}}$$

- If $T^{\mu\nu} \neq 0$,[9] one simply has $R^{(1)}_{\mu\nu} = 0$
  - Imposing a generalized "Lorenz gauge" $\bar{D}^{\nu}\bar{h}_{\mu\nu} = 0$, one has a wave equation in curved spacetime

$$\bar{D}^{\rho}\bar{D}_{\rho}\bar{h}_{\mu\nu} = 0$$

[9] Now, according to the low-frequency equation, one has $h \ll \lambda\!\!\!/ / L_B$; also, $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)^{\text{High}} = \mathcal{O}\left(h/L_B^2\right)$


THANK YOU FOR LISTENING
ANY QUESTIONS?

Thank you!