JT gravity and Island formula

July 22, 2021

Jia Tian[1] (***KITS***)

Abstract

This note is about the JT gravity and Island formula of entanglement entropy for the summer school of KITS.

[1]wukongjiaozi@ucas.ac.cn

Contents

1 Motivation **2**

2 Dimensional reduction of near extremal black hole **3**

3 Models of AdS_2 Backreaction and Holography **6**
3.1 Backreaction problem and scalar field holography 10

4 Conformal Symmetry and its breaking in two dimensional nearly anti-de-Sitter space **12**

5 Quantization of JT gravity **17**
5.1 JT gravity as a matrix integral 17
5.2 Canonical quantization of JT gravity 18

6 Information paradox in JT gravity and Island formula **24**

7 Replica wormholes **27**

8 BCFT, brane world and Doubly Holographic model **30**
8.1 Brane world 32
8.2 Doubly holographic model 34

9 Baby Universe **36**
9.1 Hawking saddles 38
9.2 Polchinski-Strominger saddles 38
9.3 Replica saddles 40
9.4 Hilbert space of baby universes and ensembles 41

A SYK model **43**
A.1 $O(N)$ Vector model 50

B Thermofield double formalism **50**

C Warped products **51**

1 Motivation

Why are we interested in JT gravity, this particular 2 dimensional gravity theory? Or what kind of questions are we going to address in this simple toy model of gravitational theory?

1. Quantum gravity. The path integral of quantum gravity with action

$$I_{EH} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} R \tag{1}$$

is potentially well defined in 2d since the dimension of the Newton constant is zero $[G_N] = 2 - d = 0$. We will see indeed that the partition function of JT gravity can be computed exactly and analytically.

2. Black hole physics. (Nearly) extremal black holes have a universal sector which is described by the JT gravity.

3. AdS/CFT duality. In 2d, the AdS_2/CFT_1 is every special and different from the higher dimensional cases. In higher dimensional cases the duality may be understood as the duality between the open and closed channels of $D-$branes. Similarly in AdS_2/CFT_1, the relevant object is D_0-brane which is point-like whose spectrum is gapped. Therefore, in the low energy limit, the only thing left is the ground state which is supposed to be described by the CFT_1. On the other hand the scale invariance requires a vanishing Hamiltonian in one dimension which means this CFT_1 is just a theory of a constraint. o go beyond the ground state, we have to zoom out a little bit from the decoupling limit to include some excitations. It also means somehow we have to break the conformal symmetry. The wormhole solutions also suggests that gravity may be not dual to a particular field theory but an ensemble average which is in tension with the well studied example AdS_5/CFT_4.

4. Information paradox. Later on we will focus on this particular problem of black hole physics to show how this problem can be solved in JT gravity. Again because JT gravity is so simple that lots of the calculation can be done explicitly.

5. Some key concepts: entanglement wedge, quantum extremal surface, Island, replica wormhole, ensemble average

2 Dimensional reduction of near extremal black hole

In this section [2], let us derive the JT gravity action from a dimensional reduction of four dimensional near extremal magnetic charged black hole. The 4D Euclidean action is

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{\hat{g}}(\hat{R} - 2\hat{\Lambda}) - \frac{1}{8\pi G}\int d^3x\sqrt{\hat{\gamma}}K^{(3)} + \frac{1}{4G}\int d^4x\sqrt{\hat{g}}F^2, \tag{2}$$

where we have included the Gibbons-Hawking boundary term in the action. The $\hat{\gamma}$ is the determinant of the induced 3D metric and $K^{(3)}$ is the trace of the extrinsic curvature of the boundary. The black hole we are considering is a magnetic one with the flux given by

$$F_{\theta\phi} = Q_m \sin\theta. \tag{3}$$

[2] We closely follow [1802.09547], "On the Dynamics of Near-Extremal Black Holes"

For dimensional reduction, we assume the 4D metric to have the form

$$ds^2 = g_{\alpha\beta}(t,r)dx^\alpha dx^\beta + \Phi^2(t,r)d\Omega_2^2, \tag{4}$$

where $g_{\alpha\beta}$ is the 2D part with coordinates (t,r) and the dilaton Φ plays the role of the radius of the 2-sphere we want to reduce. Using the identity in the appendix, we can express the 4D Ricci scalar as

$$\hat{R} = R + 2e^{-2w} - 4\nabla^2 w - 6(\nabla w)^2, \quad w = \log\Phi \tag{5}$$

and since $K^3 = \nabla^\alpha \hat{n}_\alpha$ so it is equal to $K + 2n^\alpha \partial_\alpha \Phi \Phi^{-1}$. The determined of the metric becomes

$$\sqrt{\hat{g}} = \sqrt{g}\Phi^2, \quad \sqrt{\hat{\gamma}} = \sqrt{\gamma}\Phi^2, \tag{6}$$

Other useful identities are

$$\begin{aligned} -4\nabla^2 w - 6(\nabla w)^2 &= -4(-\Phi^{-2}(\nabla\Phi)^2 + \Phi^{-1}\nabla^2\Phi) - 6\Phi^{-2}(\nabla\Phi)^2 \\ &= -4\Phi^{-1}\nabla^2\Phi - 2\Phi^{-2}(\nabla\Phi)^2, \end{aligned} \tag{7}$$

here we can integrate the first term by part as

$$\begin{aligned} &\int d^4x\sqrt{g}\Phi^2\Phi^{-1}\frac{1}{\sqrt{g}}\partial_\alpha[\sqrt{g}g^{\alpha\beta}\partial_\beta]\Phi = 4\pi\int d^2x\sqrt{g}(\nabla\Phi)^2 - 4\pi\int d^2x\partial_\alpha(\sqrt{g}g^{\alpha\beta}\Phi\partial_\beta\Phi) \\ &= -4\pi\int_{bdy}\sqrt{\gamma}\Phi n^\alpha\partial_\alpha\Phi. \end{aligned} \tag{8}$$

Combining all terms in the end we arrive at

$$\begin{aligned} S = &-\frac{1}{4G}\int d^2x\sqrt{g}\left[2 + \Phi^2(R - 2\hat{\Lambda}) + 2(\nabla\Phi)^2\right] + \frac{2\pi Q_m^2}{G}\int d^2x\sqrt{g}\Phi^{-2} \\ &-\frac{1}{2G}\int_{bdy}\sqrt{\gamma}\Phi^2 K. \end{aligned} \tag{9}$$

Next we perform a Weyl rescaling

$$g_{\alpha\beta} \to \frac{\Phi_0}{\Phi}g_{\alpha\beta} \tag{10}$$

to cancel the term $(\nabla\Phi)^2$. The useful identities are

$$\begin{aligned} &R \to e^{-2v}(R - 2\nabla^2 v), \\ &K \to e^{-v}(K + n^\alpha\partial_\alpha v), \quad e^{2v} = \frac{\Phi_0}{\Phi}. \end{aligned} \tag{11}$$

The resulted action is

$$\begin{aligned} S = &-\frac{1}{4G}\int d^2x\sqrt{g}\left[\frac{2\Phi_0}{\Phi} + \Phi^2 R - 2\Phi_0\Phi\hat{\Lambda}\right] + \frac{2\pi Q_m^2}{G}\int d^2x\sqrt{g}\frac{\Phi_0}{\Phi^3}, \\ &-\frac{1}{2G}\int_{bdy}\sqrt{\gamma}\Phi^2 K. \end{aligned} \tag{12}$$

At last we expand the dilaton around the extremal dilaton value Φ_0 as

$$\Phi = \Phi_0 + \phi. \tag{13}$$

Keeping the leading order of ϕ we arrive at

$$\begin{aligned} S = &-\frac{\Phi_0^2}{4G}\left(\int d^2x\sqrt{g}R + 2\int_{bdy}\sqrt{\gamma}K\right) \\ &-\frac{1}{2G}\int d^2x\sqrt{g}\phi(R-\Lambda_2) - \frac{1}{G}\int_{bdy}\sqrt{\gamma}\phi K + \mathcal{O}(\phi^2), \end{aligned} \tag{14}$$

which is the action of JT gravity.

Let us consider the simplest example. The metric and the electromagnetic field are given by

$$\begin{aligned} ds^2 &= -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega_2^2, \\ &= -\frac{(r-r^+)(r-r_-)}{r^2}dt^2 + \frac{r^2}{(r-r^+)(r-r_-)}dr^2 + r^2 d\Omega_2^2, \\ F &= Q\sin\theta d\phi \wedge d\theta, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \Delta &= 1 - \frac{2MG_4}{r} + \frac{Q^2 G_4}{r^2}, \quad G_4 = l_p^2, \\ r_\pm &= Ql_p + El_p^2 \pm \sqrt{2QEl_p^3 + E^2 l_p^4}, \\ E &= M - \frac{Q}{l_p}. \end{aligned} \tag{16}$$

The case $E = 0$ corresponds to the extremal black hole. To take a near horizon limit, we define a new coordinate

$$z = \frac{Q^2 l_p^2}{r - r_+}, \tag{17}$$

and take $l_p \to 0$ with z fixed. The resulting metric

$$ds^2 \sim l_p^2 Q^2 \left(\frac{-dt^2 + dz^2}{z^2} + d\Omega_2^2\right), \tag{18}$$

which is the metric of the space $AdS_2 \times S^2$. The Hawking temperature of the black hole is given by the surface gravity at the horizon:

$$T_H = \frac{r_+ - r_-}{4\pi r_+^2} \sim \frac{2\sqrt{2EQl_p^3}}{4\pi l_p^2 Q^2} = \frac{1}{2\pi}\sqrt{\frac{2E}{l_p Q^3}} + \mathcal{O}(E^{3/2}). \tag{19}$$

near extremality, where we understand this formula as the expansion of small excitation energy E. Therefore we have the energy-temperature relation for the small excitation

$$E = 2\pi^2 Q^3 l_p T_H^2. \tag{20}$$

So we can not take the near horizon limit $l_p \to 0$ with fixed E, Q and T_H. Recall in higher dimension cases the energy behaves like $E \sim VT_H$. It implies that if we fix Q then E has to be zero so there is no allowed excitation if we want to keep the near horizon geometry;

The other way to understand this conclusion is from the black hole gap. For non-extremal black hole, the energy of Hawking radiation is in the scale of the Hawking temperature T_H. So the black hole thermodynamic description will break down when $E \sim T_H$, i.e.

$$E \sim \frac{1}{l_p Q^3}, \tag{21}$$

which is called the black hole gap which gives the energy gap of the excitation above the vacuum. In the limit $l_p \to 0$ this gap is just infinity. It seems that gravitational theory with AdS_2 is boring and it seems that there is no non-trivial CFT dual.

3 Models of AdS_2 Backreaction and Holography

The main reference is [1]. The key idea is that let us consider a more general family of $1+1$ dimensional models

$$L = \frac{1}{16\pi G_N}\sqrt{-g}\{\Phi^2 R + \lambda(\nabla\Phi)^2 - U(\Phi)\}, \tag{22}$$

then for some choice of $U(\Phi)$ and the λ, the model flows from a UV completed theory to AdS_2 in the IR. The UV geometry regulates the backreaction and allows finite energy states. It turns out that the low energy (IR) dynamics is universal and can be described by a cutoff AdS_2. The cutoff not only regulates the theory but also introduces interesting dynamics.

We will focus on a simple dilaton gravity model, the AP model [1]. To study the classical solution of (14), we can ignore the topological term and boundary term for a moment but consider a more general dilaton gravity theory whose action is given by

$$\begin{aligned} S &= S_{g,\Phi} + S_{\text{matter}}, \\ S_{g,\Phi} &= \frac{1}{16\pi G}\int d^2x\sqrt{-g}\left(\Phi^2 R - U(\Phi)\right), \\ S_{\text{matter}} &= \frac{1}{32\pi G}\int d^2x\sqrt{-g}\Omega(\Phi)(\nabla f)^2, \end{aligned} \tag{23}$$

where $U(\Phi)$ is the potential of the dilaton. The 2D metric in the conformal gauge can always be written as

$$ds^2 = -e^{2w(x^+,x^-)}dx^+dx^-, \tag{24}$$

the light-cone coordinates are defined as $x^{\pm} = t \pm z$. Then the equations of motion are given by

$$\begin{aligned} 2\partial_+(e^{-2w}\partial_- e^{2w}) - \frac{1}{2}e^{2w}\partial_{\Phi^2}U(\Phi) &= (\partial_{\Phi^2}\Omega)\partial_+ f\partial_- f, \\ \partial_+(\Omega\partial_- f) + \partial_-(\Omega\partial_+ f) &= 0, \\ 4\partial_+\partial_-\Phi^2 - e^{2w}U(\Phi) &= 0, \\ -e^{2w}\partial_+(e^{-2w}\partial_+\Phi^2) &= \frac{\Omega}{2}\partial_+ f\partial_+ f, \quad (25) \\ -e^{2w}\partial_-(e^{-2w}\partial_-\Phi^2) &= \frac{\Omega}{2}\partial_- f\partial_- f. \quad (26) \end{aligned}$$

The first equation determines the metric, the second equation is the equation of motion of the matter, the third equation is the equation of motion of the dilaton and last two are constrains of the dilaton Φ. The AP model is a special case of the general dilaton model which corresponds the situation where

$$U(\Phi) = 2 - 2\Phi^2, \quad \Omega(\Phi) = 1. \tag{27}$$

In this setting, the equation of motion simplifies to

$$\begin{aligned} 4\partial_+\partial_- w + e^{2w} &= 0, \quad (28) \\ \partial_+\partial_- f &= 0, \quad (29) \\ 2\partial_+\partial_-\Phi^2 + e^{2w}(\Phi^2 - 1) &= 0, \quad (30) \\ -e^{2w}\partial_+(e^{-2w}\partial_+\Phi^2) &= \frac{1}{2}\partial_+ f\partial_- f, \quad (31) \\ -e^{2w}\partial_-(e^{-2w}\partial_-\Phi^2) &= \frac{1}{2}\partial_- f\partial_- f. \quad (32) \end{aligned}$$

Solving (28) we can determine the metric and solving (30) we can determine the dilaton. The last two equations can be thought of as the addition constraints for matters to couple with dilaton gravity. There are three static solutions

$$e^{2w} = \frac{1}{z^2}, \quad \frac{1}{\sinh^2 z}, \quad \frac{1}{\sin^2 z}, \tag{33}$$

corresponding to the Poincare path of AdS_2, a black hole with horizon at $z = \infty$ and the global AdS_2. Different solutions are related by conformal transformations. Note that the metric does not depend on the matter field at all since there is no local gravitational degrees of freedom. So let us only focus on the Poincare solution and consider the vacuum solution. Given

$$e^{2w} = \frac{4}{(x_+ - x_-)^2}, \tag{34}$$

one can solve the dilaton

$$\Phi^2 = 1 + \frac{a + b\ t + c(-t^2 + z^2)}{z} \tag{35}$$

which depends on three real parameters. The metric is invariant under the $SL(2,R)$ transformation, so not all the three parameters are physical. For example we can set them to be

$$a = 1/2, \quad b = 0, \quad c = \frac{\mu}{2}, \quad \Phi^2 = 1 + \frac{1 - \mu x^+ x^-}{x^+ - x^-}, \quad \mu > 0. \tag{36}$$

Then by performing a coordinate transformation

$$x^{\pm} = \tan X^{\pm} \tag{37}$$

we can get the global AdS_2

$$e^{2\omega} = \frac{4}{\sin^2(X^+ - X^-)}, \quad \Phi^2 = 1 + \frac{\cos X^+ \cos X^-}{\sin(X^+ - X^-)}. \tag{38}$$

Alternatively by performing a coordinate transformation

$$x^{\pm} = \frac{1}{\sqrt{\mu}} \tanh(\sqrt{\mu}(T \pm Z)), \tag{39}$$

we can get the black hole (Rindler patch) metric

$$\begin{aligned} ds^2 &= \frac{4\mu}{\sinh^2(2\sqrt{\mu}Z)}(-dT^2 + dZ^2), \\ \Phi^2 &= 1 + \sqrt{\mu}\coth(2\sqrt{\mu}Z). \end{aligned} \tag{40}$$

The place of the horizon is at

$$X^{\pm} \to \pm\infty, \quad x^{\pm} \to \pm\mu^{-1/2} \tag{41}$$

and the singularity where $\Phi^2 = 0$ is at

$$1 + \frac{1 - \mu x^+ x^-}{x^+ - x^-} = 0 \quad \to \quad (x^+ + 1/\mu)(x^- - 1/\mu) = (\mu - 1)/\mu^2. \tag{42}$$

To derive the temperature of this black hole easily, we can go to the Schwarzschild metric by performing another coordinate transformation.

$$Z = \frac{1}{2\sqrt{\mu}} \operatorname{arcCoth}\left(\frac{\rho}{\sqrt{\mu}}\right) \tag{43}$$

Then the solution reads

$$ds^2 = -4(\rho^2 - \mu)dt^2 + \frac{d\rho^2}{\rho^2 - \mu}, \quad \Phi^2 = 1 + \rho. \tag{44}$$

The Hawking temperature can be evaluated as

$$T_H = \frac{1}{4\pi}\partial_\rho \sqrt{\frac{-g_{tt}}{g_{\rho\rho}}}\Bigg|_{\rho=\sqrt{\mu}} = \frac{\sqrt{\mu}}{\pi}. \tag{45}$$

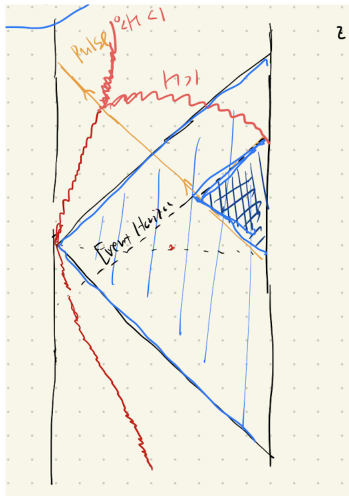

Figure 1: **different coordinates**

The Bekenstein-Hawking entropy is given by

$$S_{BH} = \frac{A}{4G_{eff}}\bigg|_{Z\to\infty} = \frac{\Phi^2}{4G}\bigg|_{\rho=\sqrt{\mu}} = \frac{1+\pi T_H}{4G}. \tag{46}$$

If there are matter fields such that T_{++} and T_{--} are not zero $T_{\pm\pm} = \partial_\pm f \partial_\pm f/16\pi G$. Then the metric is still the same while the dilaton will be given by

$$\Phi^2 = \frac{M}{x_+ - x_-}, \quad M = M_0 - I^+ + I^- \tag{47}$$

where M_0 is the sourceless solution and

$$I^\pm(x^+, x^-) = 8\pi G_N \int_{-\infty}^{x^\pm} dx'^\pm (x'^\pm - x^\mp)(x'^\pm - x^\pm) T_{\pm\pm}(x'^\pm). \tag{48}$$

One can check (47) with (48) satisfy (25) and (26). For example, let us consider a pulse of energy E,

$$T_{--} = E\delta(x^-) \tag{49}$$

then

$$\begin{aligned} I^- &= 8\pi GE \int_{-\infty}^{x^-} dx'^- (x'^- - x^+)(x'^- - x^-)\delta(x'^-) = 0, \text{ when } x^- < 0, \quad = 8\pi GEx^+x^-, \text{ when } x^- < 0 \\ &= 8\pi GEx^+x^-\Theta(x^-) \end{aligned} \tag{50}$$

Comparing with (36), we find that $\mu = 8\pi GE$.

3.1 Backreaction problem and scalar field holography

Given this explicit model, we can discuss the backreaction problem more concretely. Consider the equation of motion

$$-e^{2w}\partial_+(e^{-2w}\partial_+\Phi^2) \;=\; \frac{1}{2}\partial_+ f\partial_+ f, \tag{51}$$

We can consider this equation in an asymptotically AdS_2 (global) metric and integrate x^+ along the null line $x^- = 0$:

$$\int_0^\pi dx^+ e^{-2w}\frac{1}{2}\partial_+ f\partial_- f = \left[e^{-2w}\partial_+\Phi^2\right]|_{x^+\to 0} - \left[e^{-2w}\partial_+\Phi^2\right]|_{x^+\to\pi} > 0. \tag{52}$$

The integrand $\partial_+ f\partial_+ f$ is T_{++} so it is positive classically. Therefore the integral must give some finite positive results. However on the right hand side, if we assume an asymptotically AdS_2 space, on the line $x^- = 0$, it means

$$\begin{aligned} e^{2w} \sim \frac{1}{\sin^2 x^+} &\sim \frac{1}{x^{+2}} \quad x^+ \to 0, \\ &\sim \frac{1}{(x^+ - \pi)^2} \quad x^+ \to \pi. \end{aligned} \tag{53}$$

The non-zero result of the right hand side implies

$$\Phi^2|_{x^+=0} \sim \frac{1}{x^+}, \quad \text{or} \quad \Phi^2|_{x^+=\pi} \sim \frac{1}{x^+ - \pi}. \tag{54}$$

At least at one of the boundary, the dilaton will diverge. But recall the dilaton is actually related to the radius of S^2. So the the nonzero matter stress tensor destroys the assumed asymptotic region. Therefore to have a well defined theory in the IR we can regulate the action by adding a UV cut-off at $z = \epsilon$. So the boundary terms become

$$\begin{aligned} S_{bdy} &= \frac{1}{8\pi G}\int dt\left((-\Phi^2\partial_z w) - \int dz e^{2w} + \frac{1}{4}f\partial_z f\right), \quad e^{2w} = \frac{4}{z^2}, \\ &= \frac{1}{8\pi G}\int dt\left(-\frac{4}{\epsilon} + \frac{\Phi^2}{\epsilon} + \frac{1}{4}f\partial_z f\right) \end{aligned} \tag{55}$$

recalling the full action can be written as

$$\begin{aligned} ds^2 &= -e^{2w}dx_+dx_- \\ S &= \frac{1}{8\pi G}\int dtdz\left(\Phi^2(\partial_t^2 - \partial_z^2)\omega - e^{2w}(1-\Phi^2) - \frac{1}{4}f\partial_t\partial_t f + \frac{1}{4}f\partial_z\partial_z f\right) + \end{aligned} \tag{56}$$

The divergent pieces in (55) can be cancelled by adding proper counterterms. Therefore the regularized boundary action (which is also the generating function of the boundary field theory)is simply

$$S_{ren} = \frac{1}{32\pi G}\int dt f\partial_z f|_{z\to 0}. \tag{57}$$

At the boundary, the field is not vanishing due to its nonnormalizable mode which gives rise to the boundary source term

$$\lim_{z\to 0} f(z,t) = j(t) \tag{58}$$

such that

$$f(z,t) = \frac{1}{2\pi}\int dt' \left(\frac{1}{(z-0)+(t-t')} + \frac{1}{(z-0)-(t-t')}\right) j(t'), \tag{59}$$

$$\lim_{z\to 0} \partial_z f(z,t) = -\int dt' \frac{P}{(t-t')^2} j(t') \tag{60}$$

and

$$S_{ren} = -\frac{1}{32\pi G}\int dt dt' \frac{P}{(t-t')^2} j(t) j(t'), \tag{61}$$

where P means to take the principle part. It can not be correct since introducing a cut-off will break the conformal symmetry. Indeed the naive calculation here completely ignores the backreaction which are not negligible as we shown. Because of the backreaction the bulk time coordinate t should not be identified with the boundary time coordinate $\tilde{t}$ but we can perform a coordinate transformation to "cancel" the backreaction, see Fig. (**??**)

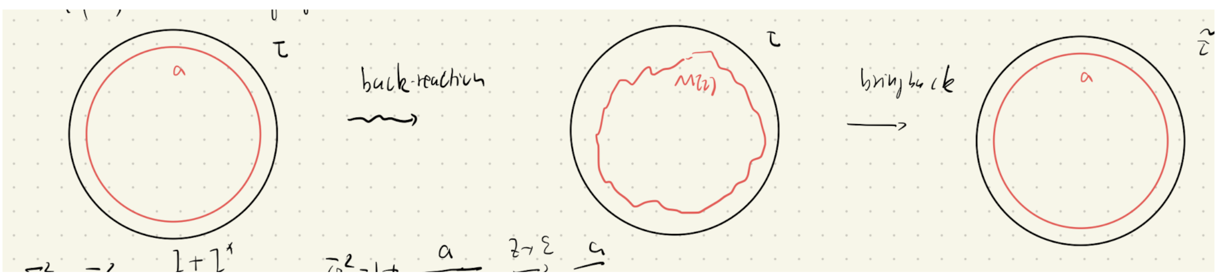

Figure 2: **backreaction**

. To coordinate transformation is derived by comparing the value of dilaton (which is a scalar) at the cut-off. From the expression (47) of dilaton with matter field, we find the relation

$$\frac{\tilde{z}(\tilde{t})}{z(t)}|_{bdy} = \frac{M_0}{M}|_{bdy}, \tag{62}$$

where $\tilde{z}(\tilde{t})|_{bdy}$ or $z(t)|_{bdy}$ define the original and deformed cut-off boundary contours. The let u to be the parameter of the boundary contour the induced metric should satisfy the cut-off condition

$$g_{uu} = \frac{(\partial_u z)^2 + (\partial_u t)^2}{z^2} = \frac{1}{\epsilon^2}, \quad \to \quad z = \epsilon \partial_u t. \tag{63}$$

Therefore the relation (62) is equivalent to

$$\frac{\tilde{z}}{z} = \frac{\partial_u \tilde{t}}{\partial_u t} = \frac{\partial \tilde{t}}{\partial t} = \frac{M_0}{M(t)}. \tag{64}$$

Integrating (64) we can find

$$t = \tilde{t} + \gamma(\tilde{t}) + \mathcal{O}(j^4), \tag{65}$$

where the expression of γ is not that important for our discussion and can be found in the original paper. Considering this correction, the generating function becomes

$$S_{ren} = -\frac{1}{32\pi G}\int d\tilde{t}d\tilde{t}' \frac{P}{(\tilde{t}-\tilde{t}')^2}(1 + \partial_t\gamma(\tilde{t}) + \partial_{t'}\gamma(\tilde{t}') - 2\frac{\gamma(\tilde{t}) - \gamma(\tilde{t}')}{\tilde{t}-\tilde{t}'})j(\tilde{t})j(\tilde{t}') + \ldots \tag{66}$$

Here I only want to illustrate the the original idea of [1] so some of calculation are not explicit. Below we will derive this generating function explicitly from the conformal symmetry breaking directly.

4 Conformal Symmetry and its breaking in two dimensional nearly anti-de-Sitter space

The main reference is [2]. The key idea is that the symmetry and the symmetry breaking govern the AdS_2 physics. Let us consider the Euclidean JT gravity with action (which is different from AP's action by a constant and Φ^2 is changed to ϕ)

$$I_{EJT} = -\frac{\phi_0}{16\pi G}\left(\int_M d^2x\sqrt{h}R + 2\int_{\partial M} K\right) - \frac{1}{16\pi G}\left(\int_M d^2x\sqrt{h}\phi(R+2) + 2\int_{\partial M}\phi_b K\right) \tag{67}$$

The Euclidean AdS_2 is just the hyperbolic disk: [INSERT Fig. 23]

$$\begin{aligned} ds^2 &= \frac{dt^2+dz^2}{z^2}, \quad t\in[-\infty,\infty], \quad \text{Poincare} && (68)\\ &= d\rho^2 + \sinh^2\rho d\tau^2, \quad \tau\in(0,2\pi), \quad \text{Rindler}, && (69)\end{aligned}$$

both of these two coordinates will cover the whole disk. For convenience, let us consider the Poincare coordinate and as before introduce a cut-off boundary which is described by the contour $(t(u), z(u))$ or $(t(u), \epsilon\partial_u t(u)$. At the boundary the dilaton diverges as

$$\phi_b = \frac{\phi_r(u)}{\epsilon}, \tag{70}$$

where $\phi_r(u)$ can be thought of as a new coupling constant (of the boundary theory). We have seen that $t(u)$ characterizes the cut-off so it also characterizes the solution space. However the global translation and rotations ($SL(2,R)$) keep $t(u)$ invariant so there is a symmetry on $t(u)$:

$$t(u) \to \frac{at(u)+b}{ct(u)+d}, \quad ad-bc=1. \tag{71}$$

Without the cut-off, the Einstein-Hilbert action in the hyperbolic space has the $t(u)$ reparameterization symmetry. Introducing cut-off spontaneously break the reparameterization symmetry down to $SL(2,R)$. Therefore we can think of the dynamics of $t(u)$ describes the Goldstone modes associated with the symmetry breaking. So the boundary theory should describe the coset $Diff(S^1)/SL(2,R)$ which is known as the Schwarzian theory. Let us now derive it from the action (68). The first term in (68) is topological so let us focus on the second term. The dilaton ϕ can be integrated out directly which simply fixes $R=-2$ and what is remaining is

$$I_{bdy} = -\frac{1}{8\pi G}\int_{\partial M}\phi_n K = -\frac{1}{8\pi G}\int du\sqrt{g_{uu}}\frac{\phi_r}{\epsilon}K = -\frac{1}{8\pi G}\int du\frac{\phi_r}{\epsilon^2}K. \tag{72}$$

The extrinsic curvature is defined by $K = g^{\mu\nu}\nabla_\mu n_\nu$. The normal vector n_μ is determined though the conditions

$$T^\mu n_\mu = 0, \quad n^\mu n_\mu = 1, \quad T^\mu = (\partial_\mu t, \partial_\mu z). \tag{73}$$

The solution is easy to obtain

$$n_\mu = \frac{1}{z\sqrt{(\partial_u t)^2 + (\partial_u z)^2}}(-\partial_u z, \partial_u t). \tag{74}$$

Therefore the extrinsic curvature can be evaluated as

$$\begin{aligned} K &= (\frac{T^\mu T^\nu}{T^2} + n^\mu n^\nu)\nabla_\mu n_\nu = \frac{T^\nu}{T^2}\nabla_T n_\nu \\ &= \frac{T^\nu}{T^2}(\partial_u n_\nu - \Gamma^\rho_{\mu\nu}n_\rho T^\mu) = \frac{t'(t'^2 + z'^2 + zz'' - zz't'')}{(t'^2+z'^2)^{3/2}} \end{aligned} \tag{75}$$

where we have used the non-vanishing connections

$$-\Gamma^t_{tz} = -\Gamma^t_{zt} = \Gamma^z_{tt} = \Gamma^z_{tz} = \frac{1}{z}, \tag{76}$$

and denoted $\partial_u t$ as t'. Next using the identity $z = \epsilon t'$ one can find

$$K = 1 + \epsilon^2 \mathrm{Sch}(t,u), \quad \mathrm{Sch}(t,u) = \frac{2t't''' - 3t''^2}{2t'^2} + \mathcal{O}(\epsilon^3) \tag{77}$$

(Note that when $u=t$, the Schwarzian term vanishes.) Substituting into the boundary action and dropping the divergent $1/\epsilon^2$ term we end up with the final boundary effective action

$$I_{bdy} = -\frac{1}{8\pi G}\int du\phi_r(u)\mathrm{Sch}(t,u). \tag{78}$$

The equation of motion can be derived from $\delta I_{bdy}/\delta u$, the result is

$$[\frac{1}{t'}\left(\frac{(t'\phi_r)'}{t'}\right)']' = 0. \tag{79}$$

Recall that in the vacuum the dilaton is given by

$$\phi = \frac{a + bt + c(t^2 + z^2)}{z} \tag{80}$$

so approaching to boundary it should be

$$\phi_r(u) = \frac{a + bt(u) + ct(u)^2}{t'(u)}. \tag{81}$$

So can check indeed this is the solution of (79). We can also absorb ϕ_r into the definition u to simplify the action further by introducing

$$d\tilde{u} = \frac{\bar{\phi}_r du}{\phi_r(u)}, \tag{82}$$

where $\bar{\phi}_r$ is some constant then we have derived the Schwarzian theory as promised

$$I_{bdy} = -C \int du \mathrm{Sch}(t, u). \tag{83}$$

The solution of the theory is

$$t(u) = \tan \frac{\pi u}{\beta} \tag{84}$$

so the period of u is β (note that $t(u)$ is equivalent to $-t(u)$). Identifying u as the thermal circle of the field theory, we can derive free energy and entropy of the theory

$$F = \log Z = -I_{bdy} = 2\pi^2 \frac{C}{\beta}, \quad S = (1 - \beta\partial_\beta) \log Z = S_0 + 4\pi^2 \frac{C}{\beta}, \tag{85}$$

which is equal to (46) (up to redefinition of C). It means that the Schwarzian action indeed captures the near extremal physics. To make direct connection to the black hole solution (Rindler patch), we introduce the Rindler circle coordinate

$$\tan \frac{\tau}{2} = t. \tag{86}$$

In terms of τ the action is given by

$$I_{bdy} = -C \int du [\mathrm{Sch}(\tau, u) + \frac{1}{2}\tau'^2] \tag{87}$$

which will also be derived directly by starting with Rindler metric of AdS_2 and compute the extrinsic curvature as we did above. In other words, τ describes the boundary fluctuation of the black hole solution (Rindler patch). We have found the classical solution this boundary theory, how about the quantization, or more explicitly can we compute the full partition function

$$Z_{Sch} = \int \frac{d\mu(\tau)}{SL(2, R)} e^{-I_{Sch}}, \tag{88}$$

where $d\mu(\tau)$ is some proper measure. Using the method of localization, it is proved in [4] the theory is 1-loop exact so we only need to study the linearized theory

$$\tau(u) = u + \epsilon(u). \tag{89}$$

The measure is still non-trivial so we will postpone the derivation. The 1-loop effective action is relative simple. Substituting the expansion directly into (87) and keep the terms up to the second order of ϵ gives the effective Lagrangian of ϵ

$$\mathcal{L} = \frac{1}{2} + (\epsilon'' + \epsilon') + \frac{1}{2}\epsilon'^2 - \frac{1}{2}\epsilon''^2. \tag{90}$$

Dropping the constant and total derivative term we end up the effective action

$$I_{eff} = \frac{C}{2}\int_0^{2\pi} (\epsilon'^2 - \epsilon''^2). \tag{91}$$

We can perform Fourier transformation

$$\epsilon = \sum \epsilon_n e^{inu} \tag{92}$$

such that

$$I_{eff} = \frac{C}{2}\sum_n (n^4 - n^2)\epsilon_n \epsilon_{-n} \tag{93}$$

after integrating over u. There are zero modes $n = 0, \pm 1$ which correspond to the $SL(2, R)$ symmetry. Given this effective action we can compute the correlation function for example

$$\langle \epsilon(u)\epsilon(0)\rangle = \frac{2}{C}\sum_{n\neq 0,\pm 1} \frac{e^{inu}}{n^2(n^2-1)} = \frac{2}{C}\oint_C \frac{ds}{e^{2\pi i s}-1}\frac{e^{isu}}{s^2(s^2-1)}, \tag{94}$$

because the integrand vanishes along the contour at infinity, the contour integral is simply given by the residues at the three poles $s = 0, \pm 1$. The result is

$$\frac{2}{C}\frac{i}{2\pi}\left((\pi - u)\sin u + \frac{5}{2}\cos u + 1 - \pi u - \frac{\pi^2}{3} - \frac{u^2}{2}\right) \tag{95}$$

We have shown the subtlety to couple JT gravity to matter field. However it is very simple to couple Schwarzian theory to a matter field with action

$$I_{\text{matter}} = \frac{1}{2}\int d^2x\sqrt{h}(h^{ab}\partial_a\chi\partial_b\chi + m^2\chi^2), \tag{96}$$

and the following asymptotic behavior

$$\begin{aligned} &\chi(z,t) = z^{1-\Delta}\tilde{\chi}_r(t) + \dots, \quad z \to 0, \\ &\Delta = \frac{1}{2}\left(1 + \sqrt{1+4m^2}\right) \end{aligned} \tag{97}$$

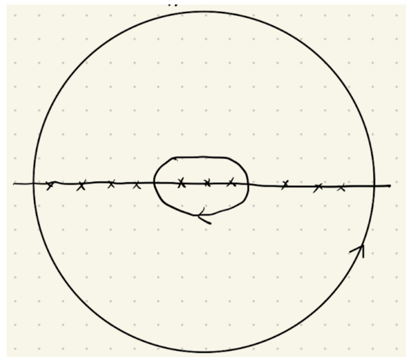

Figure 3: **integral contour**

As before, naively we will get the generating function

$$I_{gen} = -D \int dt dt' \frac{\tilde{(\chi)}_r(t)\tilde{\chi}_r(t')}{|t-t'|^{2\Delta}}. \tag{98}$$

Taking the boundary dynamics account we should expand the matter field according to the cut-off curve

$$\chi(z,t) = z(u)^{1-\Delta}\tilde{\chi}_r(t(u)) = \epsilon^{1-\Delta}t'^{1-\Delta}\tilde{\chi}_r(t(u)) \equiv \epsilon^{1-\Delta}\chi_r(u). \tag{99}$$

Therefore the generating function is

$$I_{gen} = -D \int du du' [\frac{t'(u)t'(u')}{(t(u)-t(u')^2)}]^{\Delta}\chi_r(u)\chi_r(u'), \tag{100}$$

where we can expand the kernel as with respect to ϵ by expanding $t(u)$ around the classical solution as

$$\begin{aligned} t(u) &= \tan\frac{u+\epsilon(u)}{2} \rightarrow \\ [\frac{t'(u)t'(u')}{(t(u)-t(u')^2)}]^{\Delta} &= \frac{1}{(2\sin\frac{u_{12}}{2})^{2\Delta}}[1 + B(u_1,u_2) + C(u_1,u_2) + \epsilon^{\ni}] \end{aligned} \tag{101}$$

where B and C are linear and quadratic terms of ϵ. Therefore the generating function is given by

$$\begin{aligned} -\log\langle e^{-I_{gen}}\rangle &= D\int du du' \frac{1}{(2\sin\frac{u_{12}}{2})^{2\Delta}}[1+\langle C\rangle]\chi(u)\chi_r(u') \\ +\frac{D^2}{2}\int du_1 du_2 du_3 du_4 &\frac{\chi_1\chi_2\chi_3\chi_4}{(2\sin\frac{u_{12}}{2})^{2\Delta}(2\sin\frac{u_{34}}{2})^{2\Delta}}\langle B(u_1,u_2)B(u_3,u_4)\rangle + \mathcal{O}(\epsilon^3) \end{aligned} \tag{102}$$

where $\langle C\rangle$ and $\langle BB\rangle$ can be computed in the linearized theory of Schwarzian. Some comments are in orders. We should notice that the Schwarzian theory is derived from the Euclidean Poincare AdS_2.

5 Quantization of JT gravity

There are many existing approaches of quantization of JT gravity (or Schwarzian theory):

1. Canonical quantization (dimension of phase space is 2) [3]
2. Fermionic localization (Schwarzian theory is 1-loop exact) [4]
3. Equivalence between the Schwarzian theory and a theory of a particle in a magnetic field moving in hyperbolic space [5]
4. Using $SL(2,R)$ BF (1st order formalism) theory [6]
5. JT gravity as matrix model [7]

5.1 JT gravity as a matrix integral

First let us describe the method of [7]. This approach is intuitive. The partition function is given by a path integral over all the geometries and topologies:

$$Z = \sum_{\text{topology}} \int \frac{Dg_{\mu\nu}}{\text{Diff}} (\delta(R+2)) \frac{D\tau}{SL(2,R)} e^{-I_{JT}} \tag{103}$$

$$e^{-I_{JT}} = (e^{-S_0})^{2g-2+n} e^{\sum_{i=1}^{n} \int_0^{\beta_i} du_i \text{Sch}(\tau_i, u_i)}, \tag{104}$$

so like string theory we can rewrite the partition as summation of amplitudes defined on each topology

$$Z = \sum_{g,n} Z_{g,n}(\beta_1, \dots, \beta_n)(e^{-S_0})^{2g-2+n}, \tag{105}$$

each amplitude can be denoted by a spacetime diagram for example:

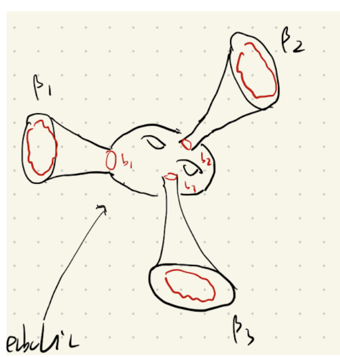

Figure 4: $\langle Z(\beta_1)Z(\beta_2)Z(\beta_2)\rangle$

Fixing the number of boundaries n and summing over g give the "correlation function" of $Z(\beta_i)$ for example

$$\langle Z(\beta_1)Z(\beta_2)Z(\beta_3)\rangle = \sum_g Z_{g,3}(\beta_1, \beta_2, \beta_3) \tag{106}$$

which can be mapped to correlation function in a random matrix model. We can always cut out one boundary with a geodesic with length b, then the geometry of this resulting boundary is Hyperbolic space with a hole inside which we will call it a trumpet geometry. Therefore the isometry of geometry space is not $SL(2,R)$ but $U(1)$ and the boundary theory now should describe the coset $\mathrm{Diff}(S^1)/U(1)$ which only differs from the Schwarzian theory by the path integral measure. Integrating out the dilaton will fix $R=-2$ so the bulk geometry must be hyperbolic then bulk integral only computes the volume of it. The volume is called the Weil-Peterson volume. In summary the amplitude is equal to

$$Z_{g,n}(\beta_1,\beta_2,\ldots,\beta_n) = \int_0^\infty b_1 db_1 \cdots \int_0^\infty b_n db_n V_{g,n}(b_1,\ldots,b_n) Z_{\mathrm{Sch}}^{\mathrm{trumpet}}(\beta_1,b_1)\ldots Z_{\mathrm{Sch}}^{\mathrm{trumpet}}(\beta_n,b_n) \tag{107}$$

The details we have omitted are the derivation of the measure bdb of Weil-Peterson volume and the derivation of the trumpet partition function $Z_{\mathrm{Sch}}^{\mathrm{trumpet}}(\beta,b)$. To derive them we have to use the 1st order formalism of JT gravity and rewrite is as a BF gauge theory then the correct measure can be computed from the symplectic form of the gauge theory. Before we move to canonical quantization let us make comments about AdS_3 gravity theory. Now the boundary is a 2D surface. If the 2D surface is the complex plane or torus then the boundary theory should describe the coset $\mathrm{Diff}(S^1)\times\mathrm{Diff}(S^1)/SL(2,R)\times SL(2,R)$. It is kind of two copies of Schwarzian theories, actually the theory is the Alekseev-Shatashvili theory. We also need the theory to describe the 3D solid trumpet which should describe $\mathrm{Diff}(S^1)\times\mathrm{Diff}(S^1)/U(1)\times U(1)$. But there is no good structure to describe the bulk integral as the Weil-Peterson volume and it seems not clear what is the proper measure like bdb.

5.2 Canonical quantization of JT gravity

Next we follow [3] to demonstrate the canonical quantization of JT gravity. Again we need to fix a patch of AdS_2, for our interests the patch is the two-sided black hole as shown in Fig.

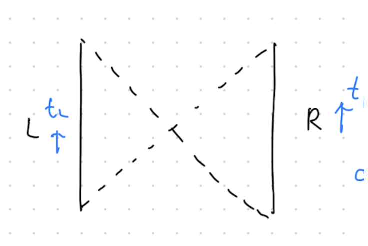

Figure 5: **Fix background**

The metric on each side is given by (40) (The constant 1 is dropped in the dilaton),

$$ds^2 = \frac{4\mu}{\sinh^2(2\sqrt{\mu}z)}(-dt^2 + dz^2),$$
$$\Phi^2 = \sqrt{\mu}\coth(2\sqrt{\mu}z). \tag{108}$$

To do canonical quantization, we need to derive the Hamiltonian which can be identified with the boundary stress tensor

$$H = \langle T_{tt}\rangle = \lim_{\epsilon\to 0}\frac{-2\epsilon}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma^{tt}} \tag{109}$$

where $\gamma_{\mu\nu}$ is the boundary metric $\gamma_{tt} = -e^{2\omega}$. The relevant term for computing the variation in the action S is

$$S = \frac{1}{16\pi G}\int d^2x\sqrt{-g}\Phi^2 R - \frac{1}{8\pi G}\int dt\sqrt{-\gamma}\Phi^2 K, \quad K = \gamma^{\mu\nu}\nabla_\mu n_\nu. \tag{110}$$

If Φ^2 is a constant then $\delta S/\delta\gamma_{ab}$ will lead to the usual result of general relativity

$$K^{\mu\nu} - K\gamma^{\mu\nu} = 0. \tag{111}$$

The non-vanishing term comes from the variation $\delta\partial_z g_{tt}$ in the bulk since

$$\int d^2x f(x)\delta\partial_z g_{tt} \to \int d^2x\partial_z(f(x)\delta g_{tt}) \sim \int dt f(x)\delta\gamma_{tt}. \tag{112}$$

and the terms from the counterterm which are supposed to cancel the divergence:

$$S_{ct} = \int dt\sqrt{-\gamma}\left(\frac{1}{8\pi G}(-\Phi^2)\right) \tag{113}$$

Considering the identity

$$\sqrt{g}g^{AB}\delta R_{AB} = \partial_C(\sqrt{g}g^{AB}\delta\Gamma^C_{AB}) - \partial_B(\sqrt{g}g^{AB}\delta\Gamma^C_{AC}) \tag{114}$$

Integrating by parts leads to

$$\partial_B\Phi^2\sqrt{g}g^{AB}\delta\Gamma^C_{AC} - \partial_C\Phi^2(\sqrt{g}g^{AB}\delta\Gamma^C_{AB}), \tag{115}$$

the component will survive at the boundary is g^{tt} and only non-vanishing term which contain $\partial_z g_{tt}$ is

$$\partial_z\Phi^2\sqrt{g}\gamma^{tt}\gamma^{zz}\delta\partial_z g_{tt}. \tag{116}$$

Therefore we have

$$\frac{\delta S}{\delta\gamma_{tt}} = \frac{1}{16\pi G}\partial_z\Phi^2\gamma^{tt}\gamma^{zz}\sqrt{g},$$
$$\frac{\delta S}{\delta\gamma^{tt}} = \sqrt{g}\gamma_{tt}\partial_z\Phi^2 = \frac{1}{16\pi G}e^{2w}\partial_z\Phi^2 \tag{117}$$

and the unnormalized hamiltonian is

$$\frac{\epsilon}{8\pi G} e^{w} \partial_z \Phi^2 + \epsilon e^{2w} \frac{\Phi^2}{8\pi G}. \tag{118}$$

The asymptotic expansions of Φ^2 and e^w are

$$e^w = \frac{1}{z} - \frac{2u}{3} + \mathcal{O}(z), \quad \Phi^2 = \frac{1}{2z} + \frac{2uz}{3} + \mathcal{O}(z^2) \tag{119}$$

so the finite piece of (118) is the final hamiltonian is

$$H = H_L + H_R = 2\frac{\mu}{8\pi G} \equiv 2\frac{\Phi_h^2}{\Phi_b}, \quad \Phi_b = 8\pi G. \tag{120}$$

In the Hamiltonian (120), there is only one dynamical variable Φ_h. There must be another variable which conjugate to it. Intuitively the conjugation of energy should the related to time so the guess would be

$$\delta = t_L + t_R \tag{121}$$

which measures the relative time shift between the two boundaries. Thus we arrived at the 2D Hamiltonian system:

$$\dot{\delta} = 1, \quad \dot{H} = \dot{\Phi}_h = 0, \quad \omega = d\delta \wedge dH. \tag{122}$$

However this Hamiltonian does not lead to a sensible Schrodinger equation because (120) only depends on Φ_h but not its conjugate. So we can perform a canonical transformation. The variable which has geometric meaning is the geodesic length between the two boundaries with $t_L = t_R$. To compute this length, we need to introduce the static coordinates

$$ds^2 = -(1+x^2)d\tau^2 + \frac{dx^2}{1+x^2}, \tag{123}$$

$$\Phi = \Phi_h \sqrt{1+x^2} \cos\tau \tag{124}$$

which is related to the Schwarzschild coordinate (87) though

$$\begin{aligned} \sqrt{1+x^2}\cos\tau &= \rho/\sqrt{\mu} \\ \sqrt{1+x^2}\sin\tau &= \sqrt{(\rho/\sqrt{\mu})^2 - 1}\,\sinh(2\sqrt{\mu}t) \\ x &= \sqrt{(\rho/\sqrt{\mu})^2 - 1}\,\cosh(2\sqrt{\mu}t) \end{aligned} \tag{125}$$

In particular, the bulk time τ is related to boundary time via

$$x, \rho \to \infty, \quad \cos\tau = \frac{1}{\cosh(2\sqrt{\mu}t)} = \frac{1}{\cosh(\sqrt{\mu}\delta)}. \tag{126}$$

Therefore the geodesic length is given by

$$L_0 = 2\int_0^{x_c} \frac{dx}{\sqrt{1+x^2}} = 2Arc\sinh(x_c) = 2\log(2x_c), \quad x_c \to \infty \tag{127}$$

But the distance is divergent so, we define the renormalized geodesic length

$$
\begin{aligned}
L \equiv L_0 - 2\log(2\Phi\,|_{bdy}) &= 2\log\frac{x_c}{\rho_c} = 2\log\frac{\cosh(\sqrt{\mu}\delta)}{\sqrt{\mu}} \\
&= 2\log\left(\cosh(\sqrt{\frac{\phi_b H}{2}}\delta)\right) - \log\frac{\phi_b H}{2}.
\end{aligned} \tag{128}
$$

Its conjugation is

$$
P = \sqrt{\frac{2H}{\phi_b}}\tanh\left(\sqrt{\frac{H\phi_b}{2}}\delta\right). \tag{129}
$$

It is easy to check that

$$
\frac{\partial L}{\partial \delta}\frac{\partial P}{\partial H} - \frac{\partial L}{\partial H}\frac{\partial P}{\partial \delta} = 1. \tag{130}
$$

Now we can solve the Hamiltonian in terms of new canonical coordinates

$$
H = \frac{P^2}{2\phi_b} + \frac{2}{\phi_b}e^{-L}, \tag{131}
$$

then the energy eigenstates $\langle l|E\rangle = \psi_E$ should be determined from the Schrodinger equation

$$
-\frac{1}{2\phi_b}\psi_E''(L) + \frac{1}{\phi_b}e^{-L}\psi_E(L) = E\psi_E(L), \tag{132}
$$

it just describes the mechanics of a non-relativistic particle moving in an exponential potential. The solutions are given by the modified Bessel functions and the complete basis of the wavefunction is

$$
\psi_E(L) = \langle l|E\rangle = 4K_{2i\sqrt{2E}}(4e^{-l/2}), \quad \rho(E) = \frac{1}{2\pi^2}\sinh(2\pi\sqrt{2E}) \tag{133}
$$

satisfying

$$
\begin{aligned}
&\int_{-\infty}^{\infty} dl\langle E|l\rangle\langle l|E'\rangle = \frac{\delta(E-E')}{\rho(E)}, \\
&\int_0^E dE\rho(E)\langle l|E\rangle\langle E|l'\rangle = \delta(l-l')
\end{aligned} \tag{134}
$$

where $\rho(E)$ is the density of state. Even though we have quantized JT gravity from the perspective of a non-relativistic quantum particle, but JT is a gravity theory we also want to know the relation between the geometry and quantum states. In particular, we may want to do how the Euclidean path integral relates to the quantum states.

The idea is that since the geometry is a two-sided black hole, then the Euclidean path integral should prepare the thermofield double state such that

$$
\mathrm{Tr}[e^{-\beta H}] = \mathrm{Tr}[e^{-\beta(H_L+H_R)}]\langle TFD_\beta|TFD_\beta\rangle. \tag{135}
$$

From the general AdS/CFT dictionary we expect the thermofield double state is dual to the Hartle-Hawking state HH_β. Therefore the wavefunction $\langle l|HH_\beta\rangle$ equals to the Euclidean path integral over geometries with the topology of a disk and boundaries consisting of an asympotically AdS portion of length $\beta/2$ and a geodesic of length l:

Figure 6: **HH state**

$$\psi_{D,\beta/2}(l) = \langle l|HH_\beta\rangle = \int_0^\infty dE\rho(E)e^{-\frac{\beta}{2}E}\psi_E(l) \tag{136}$$

The overlap of Hartile-Hawking wavefunction computes the disk partition function as expected

$$\langle HH_\beta|HH_\beta\rangle = \int_{-\infty}^{\infty} dl\psi^\star_{D,\beta/2}\psi_{D,\beta/2}. \tag{137}$$

From the point of view of quantum gravity, we may define a boundary operator

$$\begin{aligned}&(\hat{\psi}_{D,\beta/2}, \hat{\psi}_{D,\beta/2}),\\ &|HH_\beta\rangle = \hat{\psi}_{D,\beta/2}|HH\rangle \equiv |\psi_{D,\beta/2}\rangle.\end{aligned} \tag{138}$$

then (137) simply computes the expectation value of this operator. Apart from the disk geometry, there is also the trumpet geometry. The trumpet has another boundary which is closed geodesic with length b, we may also associate a operator $\hat{b}$ to it. Therefore, the path integral over the trumpet geometry should be equal to

$$\langle HH|(\hat{\psi}_{D,\beta/2}, \hat{\psi}_{D,\beta/2})\hat{b}|HH\rangle \tag{139}$$

which can be rewritten as

$$\langle\psi_{D,\beta/2}|\psi_{Tr,\beta/2,b}\rangle, \quad |\psi_{Tr,\beta/2,b}\rangle = \hat{\psi}_{D,\beta/2}\hat{b}|HH\rangle, \tag{140}$$

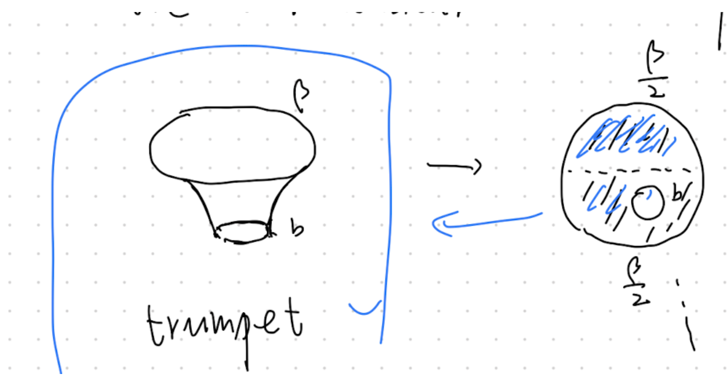

Figure 7: **trumpet**

on the $|l\rangle$ basis, the trumpet wavefunction is given by

$$\langle l|\psi_{Tr,\beta/2,b}\rangle = \int_0^\infty dE \frac{\cos(b\sqrt{2E})}{\pi\sqrt{2E}} e^{-\frac{\beta}{2}E}\langle l|E\rangle. \tag{141}$$

The missing ingredient is how to understand Weil-Peterson volume from the point of view quantum states. Schematically the volume $V_{g,3}(b_1, b_2, b_3)$ can be written as

$$\langle HH|\hat{b}_1\hat{b}_2\hat{b}_3|HH\rangle_g, \tag{142}$$

which does not depend on the asymptotical boundary at all so we expect it should be described by 2D topological gravity only and we know it is corresponding to the integral of moduli space. A possible approach to study it explicitly is using the topological BF gauge theory, where the moduli space of gravitational theory is related to the moduli space of flat connection of gauge theory. However how to do the canonical quantization for JT gravity with more than two asymptotical boundaries and taking other possible topology into account directly as far as I know has not been well studied. We finish this section with some comments:

1. The eigenstates from the canonical quantization do not factorize as $H_L \otimes H_R$. In particular the quantum theory is not equal to two copies of Schwarzian theories as we naively expected.
2. The matrix integral quantization of JT gravity suggests the dual theory of JT gravity is not a explicit theory but an ensemble average of a family of theories.
3. The wormholes (configurations with more than one asymptotical boundaries) are not classical solutions of JT gravity so they are not saddle points of the Euclidean path integral. Should we include them into the path integral is still a open problem. Including them may cause the factorization puzzle: $\langle Z\rangle^2 \neq \langle ZZ\rangle$ which prefers the ensemble average interpretation i.e. the Euclidean path integral does not compute the exact values but some average over an ensemble. But the ensemble average interpretation is in tension with

the AdS/CFT. Another possibility is that all the wormhole contributions will cancel each other. This is also unlikely because we have seen that the wormhole needs to be included in the computation of some physical quantities. Perhaps they can be understood as complex saddles or constrained instantons. We need more thoughts to understand the so called Wormhole-paradigm.

6 Information paradox in JT gravity and Island formula

For a pedagogical review without technical details see [8] where the information paradox and island formula are discussed in general. Here we focus on the toy model: JT gravity. First let us formula a version of information paradox in JT gravity. The main reference is [9]. The key idea is that AdS_2 black hole is eternal, in other words, it does not evaporate so we couple the black hole at nonzero temperature to a bath which is described by a non-gravitional flat spacetime.

The whole geometry is shown in Fig. (6)

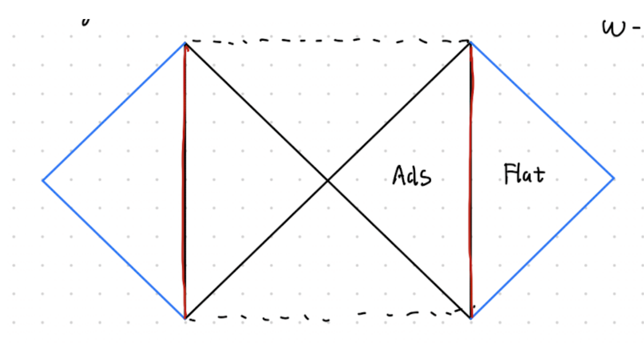

Figure 8: **whole geometry**

The global coordinate is $w^{\pm}$ which are related to left and right-coordinates via

$$w^{\pm} = \pm e^{\pm 2\pi y_R^{\pm}/\beta}, \quad w^{\pm} = \mp e^{\mp 2\pi y_L^{\pm}/\beta},$$
$$y_L^{\pm} = t \mp z, \quad y_R^{\pm} = t \pm z. \tag{143}$$

Each side is also divided into two regions: AdS and flat $bath$ and the corresponding metrics are (focus on the right side)

$$ds^2_{ads} = -\frac{4\pi^2}{\beta^2}\frac{dy^+dy^-}{\sinh^2\frac{\pi}{\beta}(y^- - y^+)}, \quad ds^2_{bath} = -\frac{1}{\epsilon^2}dy^+dy^- \tag{144}$$

where we have assumed the AdS has a cut-off at $z = -\frac{1}{\epsilon}$ and the scale factor $1/\epsilon^2$ in the ds^2_{bath} guarantees that these two metrics agree at the cut-off. The dilaton only exists in the AdS region and has the profile (we have restored the extremal term ϕ_0)[3]

$$\phi = \phi_0 + \frac{2\pi\phi_r}{\beta}\frac{1}{\tanh\frac{\pi}{\beta}(y^- - y^+)} = \phi_0 + \frac{2\pi\phi_r}{\beta}\frac{1-w^+w^-}{1+w^+w^-}. \tag{145}$$

For simplicity we assume the matter field is described by a CFT and the Hawking modes are collected in the bath (because they contain the global infinities) so we would like to compute the entanglement entropy $S_{(-\infty,-b)\cup(b,\infty)}$associated with interval $(-\infty,-b)\cup(b,\infty)$.

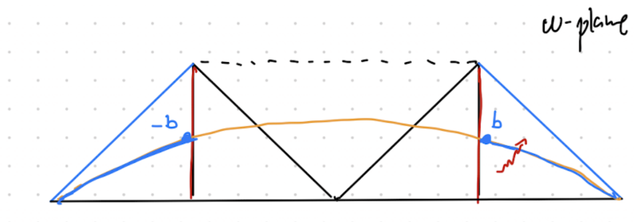

Figure 9: **no island**

Assuming the state on the whole Cauchy surface is pure then $S_{(-\infty,-b)\cup(b,\infty)} = S_{(-b,b)}$ which is just the entanglement entropy of a single interval. The general formula is [4]

$$S_{(x_1,x_2)} = \frac{c}{6}\log\left(\frac{-(\omega_{12}^+\omega_{12}^-)}{\Omega_1\Omega_2}\right) \tag{146}$$

where the conformal factor can be computed through the relations (143)

$$(\Omega_1\Omega_2) = \sqrt{\partial_{y^+}w_1^+\partial_{y^-}w_1^-\partial_{y^+}w_2^+\partial_{y^-}w_2^-} = \left(\frac{2\pi}{\beta}\right)^2 e^{\frac{4\pi}{\beta}z} \tag{147}$$

since the w_1 and w_2 are in the bath region where y are the proper coordinates. Therefore the entanglement entropy is

$$S_{(-b,b)} = \frac{c}{6}\log\left(\frac{(e^{-\frac{2\pi}{\beta}y_L^+}+e^{\frac{2\pi}{\beta}y_R^+})(e^{\frac{2\pi}{\beta}y_L^-}+e^{-\frac{2\pi}{\beta}y_R^-})}{\left(\frac{2\pi}{\beta}\right)^2 e^{\frac{4\pi}{\beta}z}}\right) = \frac{c}{3}\log\left(\frac{\beta}{\pi}\cosh\frac{2\pi t}{\beta}\right) \tag{148}$$

and the late time $t >> \beta$ it grows linearly in time

$$S_{(-b,b)} \sim \frac{c}{3}\frac{2\pi t}{\beta}. \tag{149}$$

[3]Note that $z<0$ in AdS comparing with (108)

[4]The UV cut-off factor ϵ is dropped.

However the black hole only has finite amount of degrees of freedom to entangle with the matter field because of the Bekenstein-Hawking entropy bound $2S^{\beta}_{BH} = 2(\phi_0 + \frac{2\pi\phi_r}{\beta})$. Therefore the linear growth of the entanglement entropy in the late time is contradictory. Where is our mistake? The only possible mistake is the entropy formula (146) we used. When we derive this formula with replica trick we fix the background and even in the replica geometry we only consider one particular background. However if the background is dynamical, the correct semi-classical approximation should include all the possible saddle geometries (perhaps with complex saddles and constrained instantons). It was proposed in [12, 13, 14], the correct formula of the entropy of Hawking radiation is

$$S[\mathbf{Rad}] = \min\{\text{ext}[S[\text{Rad}] \cup I + \frac{\text{Area}[\partial I]}{4G_N}]\} \tag{150}$$

where I is called island and the extremal $\partial I \equiv \Sigma$ is a co-dimensional two surface, known as the Quantum Extremal Surface (QES). In the context of JT gravity, the QES is just a point and the Area$[\partial I]$ is the value of the dilaton at the QES. Let us see how this formula resolves the information paradox first then "derive" this formula from different perspectives.

For simplicity we set $b = 0$, therefore the two points p_2 and p_4 in the bath are at

$$w_2^{\pm} = w_4^{\mp} = \pm e^{\pm 2\pi t/\beta}. \tag{151}$$

Assume that there is an island in AdS with QES at p_1 and p_3 at $w_1^{\pm} = w_3^{\mp}$ as shown in Fig. (8)

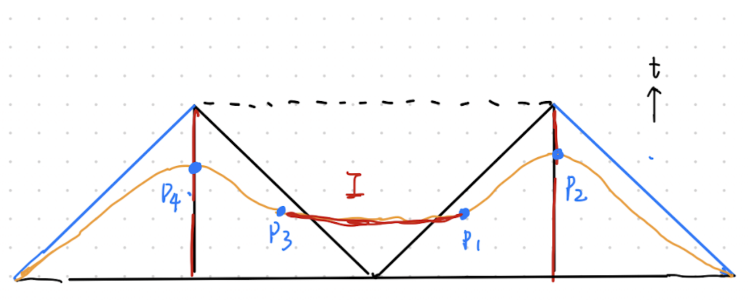

Figure 10: **with island**

Now we need compute the entanglement entropy associated with two intervals. In the late time, p_2 and p_4 approach ∞ then we expect that these two intervals are very far way so the contribution from the left and the contribution from the right decouple such that

$$S[\mathbf{Rad}] = 2 \times \left(\frac{\phi(w_1^{\pm})}{4G_N} + \frac{c}{6} \log(\frac{-w_{12}^2}{\Omega_1 \Omega_2}) \right), \tag{152}$$

where p_1 is in the AdS and p_2 is in the bath so the conformal factor is

$$\Omega_1^{-2} = \frac{4}{(1+w_1^+ w_1^-)^2}, \quad \Omega_2^{-2} = \left(\frac{\beta}{2\pi}\right)^2. \tag{153}$$

Therefore the entanglement entropy is

$$S[\mathbf{Rad}] = \frac{\phi_0}{2G_n} + \frac{c}{3}\log\frac{\beta}{\pi} + \frac{c}{3}\log\frac{-w_{12}^2}{1+w_1^+ w_1^-} + a\frac{1-w_1^+ w_1^-}{1+w_1^+ w_1^-}, \tag{154}$$

$$a \equiv \frac{\pi}{\beta}\frac{\phi_r}{G_N}. \tag{155}$$

The extremal conditions are

$$\partial S/\partial w_1^+ = \partial S/\partial w_1^- = 0. \tag{156}$$

These are third order algebraic equations then solutions are easy to obtain but somehow complicated. However in the semi-classical limit $G_N \to 0$ the solutions are simplified. There are three possible solutions

$$1. \quad w_1^\pm = w_2^\pm, \tag{157}$$

$$2. \quad w_1^\pm = \frac{6a}{c}\frac{1}{w_2^\mp}, \tag{158}$$

$$3. \quad w_1^\pm = -\frac{c}{6a}\frac{1}{w_2^\mp}, \tag{159}$$

(157) is just the trivial solution without island. (158) is not acceptable because it implies the p_1 and p_2 are on the different side but we have assumed that p_1 and p_2 are both in the right side. One can also check that (158) leads to a complex entropy so the solution does not correspond to a QES. The solution (159) is physical and corresponds to a new QES. Substituting (159) into (154) gives

$$S[\mathbf{Rad}] = 2\times(\frac{\phi_0}{4G_N} + \frac{2\pi}{\beta}\frac{\phi_r}{4G_N}) + \frac{c}{3}\log\frac{\beta}{\pi} = 2S_{BH} + \frac{c}{3}\log\frac{\beta}{\pi} \tag{160}$$

which is a constant. Note that island formula also gives a quantum correction of order $\mathcal{O}(G_N^0)$.

7 Replica wormholes

To describe replica wormholes, let us consider the toy model: JT gravity with an end of world brane (EOW brane). To further simplify the model, we do not treat EOW dynamically. The only role they play is to provide new boundary conditions. We mainly follow [10].

Recall that JT gravity originally has the asymptotical boundary operator

$$(\hat{\psi}_{D,\beta/2}, \hat{\psi}_{D,\beta/2}), \tag{161}$$

which corresponds to a loop with renormalized length β. The EOW brane introduces new boundary operator

$$(\psi_i, \psi_j) \tag{162}$$

which corresponds to an interval and Euclidean path integral computes the inner product

$$\frac{\langle HH|(\psi_i, \psi_j)|HH\rangle}{\langle HH|HH\rangle} = \langle \psi_i|\psi_j\rangle = \delta_{ij}, \quad i, j = 1, 2, \ldots, k. \tag{163}$$

Figure 11: **new boundary operator**

To model an evaporating black hole, we use the brane state $|\psi_i\rangle$ to describe the black hole quantum state (which is also the state of the partner of Hawking radiation) and couple to it with auxiliary system R which models the Hawking radiation. So the state of the whole system is

$$|\Psi\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} |\psi_i\rangle|i\rangle_R. \tag{164}$$

We are interested in the entanglement entropy of the radiation so we consider the reduced density matrix

$$\sum_k \langle \psi_k|(\frac{1}{k} \sum_{i,j} |i\rangle\langle j| \otimes |\psi_i\rangle\langle\psi_j|)|\psi_k\rangle = \rho_R,$$
$$\rho_R = \frac{1}{k} \sum_{i,j=1}^{k} |j\rangle\langle i|_R \langle \psi_i|\psi_j\rangle. \tag{165}$$

If there is no island the entropy is $\log k$ while the new QES should be close to the horizon and gives the entropy S_{BH} so we expect the entropy of the radiation is

$$S(R) = \min\{\log k, S_{BH}\}. \tag{166}$$

Again the entropy will be computed using the replica trick

$$S_R = -\mathrm{Tr}(\rho_R \log \rho_R) = -\lim_{n\to 1} \frac{1}{n-1} \log \mathrm{Tr}(\rho_R^n) \tag{167}$$

and as a simple illustration of replica wormhole we will consider the simple quantity: the purity $\mathrm{Tr}(\rho_R^2)$. The key point is that we should use Euclidean path integral to compute it directly instead of computing ρ_R first. Using (162) we can write the reduced density matrix and its square as

$$\rho_R = \frac{1}{k}\sum_{i,j=1}^{k} |j\rangle\langle i|\langle HH|(\psi_i,\psi_j)|HH\rangle\langle HH|HH\rangle^{-1}, \tag{168}$$

$$\rho_R^2 = \frac{1}{k^2}\sum_{i,j,k} |i\rangle\langle k||k\rangle\langle j|\langle HH|(\psi_i,\psi_k)(\psi_k,\psi_j)|HH\rangle\langle HH|HH\rangle^{-2}. \tag{169}$$

There are four intervals so there are two possible ways to connect them to form closed boundary so

$$\langle HH|(\psi_i,\psi_k)(\psi_k,\psi_j)|HH\rangle = \delta_{ik}\delta_{kj}\langle HH|HH\rangle^2 + \delta_{kk}\delta_{ij}\langle HH|HH\rangle. \tag{170}$$

Therefore the purity is

$$\mathrm{Tr}(\rho_R^2) = \frac{1}{k^2}\sum_{i,j,k}\delta_{ij}\delta_{kk}(Z_0^2\delta_{ik}\delta_{jk} + Z_0\delta_{kk}\delta_{ij})Z_0^{-2} = \frac{1}{k} + \frac{1}{Z_0} \sim \frac{1}{k} + e^{-S_{BH}}, \tag{171}$$

where $Z_0 \equiv \langle HH|HH\rangle$. If k is small then the first them (disconnected geometry) dominates while if k is very larger then the second term (wormhole geometry) dominates. However our calculation seems to be contradictory. The reduced density matrix is

$$\rho_R = \frac{1}{k}\sum_i |i\rangle\langle i| \tag{172}$$

and then we should have

$$\rho_R^2 = \rho_R\rho_R = \frac{1}{k^2}\sum_i |i\rangle\langle i| \tag{173}$$

which is different from the path integral result. The mistake comes from the orthogonal condition (163). When we define the quantum states $|\psi_i\rangle$ of EOW, we have to fix the AdS background to do the quantization such that

$$\langle\psi_i|\psi_j\rangle = \int dE\rho(E)\Psi_{D,i}\Psi_{D,j} = \delta_{ij}, \tag{174}$$

where $\Psi_{D,i}$ are the energy eigenfunctions. As we have shown in last section, because other topologies will also contribute to this inner product so we also need the eigenfunction correspond to the trumpet geometry. The exact inner product schematically can be written as

$$\langle\psi_i|\psi_j\rangle = \delta_{ij} + (e^{-S_0})^{2g-2+1}\sum_g\int dbV_g(b)\int dE\mu(E,b)\Psi_{D,i}\Psi_{Tr,j}(b). \tag{175}$$

The details of the quantization of JT gravity with EOW can be found in [11]. Since we do not know how to do the canonical quantization directly for gravitational theory and what we have is the Euclidean path integral. One suggestion to resolve the contradiction is to interpret our naive path integral results as some ensemble average:

$$\overline{\langle\psi_i|\psi_j\rangle} = \delta_{ij}, \quad \overline{|\psi_i|\psi_j|^2} = \delta_{ij} + e^{-S_{BH}}. \tag{176}$$

This interpretation also suggests that JT gravity theory (with EOW) is dual to some ensemble averaged theory (for example the matrix model).

Even though the replica wormhole makes a lot of sense but we have not proved (8). The new entropy formula (8) resembles the RT formula. Indeed we can follow Lewkowycz and Maldacena's derivation of RT formula to derive to derive (8) [15]. The basic idea is very straightforward that we should also integrate over the gravity when we use the replica trick to compute the Renyi entropy:

$$e^{-(n-1)S_A^{(n)}} = \int_{M_n} DgD\phi e^{-S_{grav}[g]-S_{QFT}[g,\phi]} \tag{177}$$

then we consider the quotient manifold $\tilde{M} = M_n/Z_n$. In $\tilde{M}$ at the fixed points which are co-dimension two surfaces ∂A of the quotient Z_n there will be conical singularities. To support these singularities in our gravity theory we can insert cosmic branes with a tensor $T = 1-n$ to the fixed points. Therefore the gravity action should be modified to

$$S_{grav} \to S_{grav} + \frac{1-n}{4G}|\partial A|, \tag{178}$$

such that the area dependence terms manifest.

8 BCFT, brane world and Doubly Holographic model

The discussion in this section is schematic and conceptual. The details can be found in [16] and [13]. The key idea is that we assume the matter CFT is also holographic then the whole system $S_{JT}+S_{matter}$ can be described by a AdS_3 gravity theory with a dynamical boundary so that we can think of the system is holographically dual to a 2D BCFT. Then the claim is that the island formula is just the holographic entanglement entropy of a BCFT [16].

Introducing boundaries to CFT will break conformal symmetries, however it is possible by choosing proper boundary conditions such that the conformal symmetry $SO(2,d)$ is only broken down to $SO(2,d-1)$. Such boundary conditions are also described by so-called conformal boundary states or Cardy states. For a pedagogical introduction of BCFT, see the textbooks [17] and [18]. The AdS/CFT duality can also be generalized to AdS/BCFT duality: CFT on a manifold A with a boundary ∂A is dual to Gravity on an asymptotically AdS space M, $\quad \partial M = A \cup Q$ with Q is some codimension-1 surface.

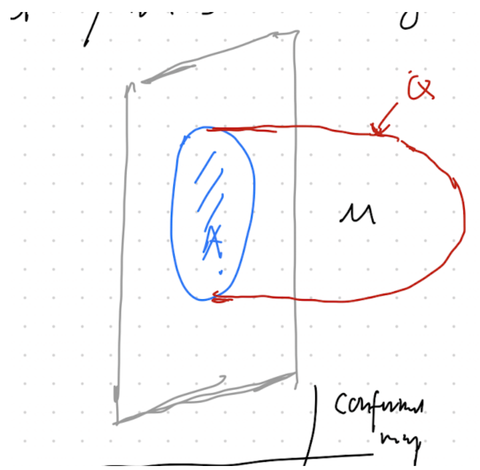

Figure 12: **BCFT/AdS**

We can think of that the boundary ∂A extends to the bulk. Let us consider the simple model $AdS_3/BCFT_2$ to “derive” the island formula. Assume that the BCFT is free and consider an interval $A = (a, b)$. If there is no boundary, to compute the entanglement entropy with the replica method we basically need to compute two point function

$$\mathrm{Tr}(\rho_A^n) = \langle \mathcal{T}_n(a)\mathcal{T}_n(b)\rangle, \tag{179}$$

or holographically compute the length of geodesic connection a and b. However after introducing the boundary which for simplicity we assume it perpendicularly extends to the bulk, we have to use the method of image (because the boundary will change the propagator) to compute the two point function such that effectively we are computing a four point function

$$\langle \mathcal{T}_n(-a)\mathcal{T}_n(-b)\mathcal{T}_n(a)\mathcal{T}_n(b)\rangle, \tag{180}$$

which can also be computed holographically. Now there are two possible configurations: the RT surface may intersect with the boundary in the bulk or RT surface does not intersect with the boundary in the bulk. So the RT formula taking account into the boundary should be

$$S_A = \min_{\gamma_A}\mathrm{Ext}\left(\frac{\mathrm{Area}(\gamma_A)}{4G_{N,d=3}}\right), \quad \partial\gamma_A = \partial A \cup \partial B \equiv S_{A\cup B}, \tag{181}$$

where the intersection B is the analogue of island. But (181) is not exactly equal to (8). The missing piece is $\frac{\mathrm{Area}(\partial B)}{4G_{N,d=2}}$ which comes from a gravity contribution. This motivates us to make the boundary to be dynamical. In other words, we should expect the boundary should be described by a gravitational theory. Gravity with also gravitational boundary is captured by the Randall-Sundrum (RS) gravity or brane gravity. Before introducing the RS brane, let us add more details to the formulation of $AdS/BCFT$. The gravity action is given by

$$S = \frac{1}{16\pi G_N} \int_M \sqrt{-g}(R - 2\Lambda) + \frac{1}{8\pi G} \int_Q \sqrt{-h}(K - T), \tag{182}$$

where the constant T is the tension of the boundary or we can think of we add some boundary matter field whose stress-energy tensor is $T_{ab} = -Th_{ab}$. The crucial point is that we allow the boundary metric to fluctuate (equivalently we choose the Neumann boundary condition on Q). The variation of the boundary metric will lead to

$$K_{ab} = (K - T)h_{ab}, \quad \rightarrow \quad K = \frac{d}{d-1}T, \tag{183}$$

which will fix the position of Q. Let the metric of AdS_{d+1} to be

$$ds^2 = d\rho^2 + \cosh^2 \frac{\rho}{R} ds^2_{AdS_d}. \tag{184}$$

If we put Q at $\rho = \rho_\star$ then the extrinsic curvature on Q can be computed as

$$K_{ab} = \frac{1}{2}\frac{\partial g_{ab}}{\partial \rho} = \frac{1}{R} \tanh \frac{\rho}{R} h_{ab}, \tag{185}$$

where g_{ab} and h_{ab} are the metric of AdS_{d+1} and AdS_d, respectively. Therefore (183) leads to

$$T = \frac{d-1}{R} \tanh \frac{\rho_\star}{R}. \tag{186}$$

Recall the AdS_{d+1} boundary is at $\rho = -\infty$.

Even though we kind of have a brane Q with tensor T but there is no gravity on Q yet. This action (182) is an analogue of JT gravity with EOW brane. To promote EOW brane to a gravitational brane we can use the RS brane world construction or add an intrinsic gravity term in the brane action (this scenario is called the Davli-Gabadadze-Porrati (DGP) gravity).

8.1 Brane world

The main reference is [19]. Let us look at the Hilbert-Einstein action of gravity

$$S = \frac{1}{16\pi G_{d+1}} \int_M \sqrt{-g}(R + \frac{d(d-1)}{L^2}) + \frac{1}{8\pi G_{d+1}} \int_{\partial M} \sqrt{-\gamma}K, \tag{187}$$

in which each term is divergent because both the bulk volume and boundary volume are infinite. So in AdS/CFT calculations, a series of boundary counterterms have to be added to the action to make the action well defined. Usually the boundary is chosen to be asymptotical boundary since the bulk metric is determined locally by the conformal structure of the asymptotical boundary up to very high order. In the brane world construction, this regulator surface is replaced by the brane located at some finite radius and the divergent terms become the gravitational action of the brane theory. The gravitational action is determined through two steps:

1. Start from AdS_{d+1} gravity and consider a Fefferman-Graham expansion near the boundary of an asymptotic AdS_d geometry;
2. Integrating the bulk gravity action over the radial direction to the regulator surface.

For example when $d > 2$, these divergent terms read

$$I_{\text{di}} = \frac{1}{16\pi G_{d+1}} \int d^d x \sqrt{-\tilde{g}} \left[\frac{2(d-1)}{L} + \frac{L}{2(d-2)} \tilde{R} + \frac{L^3}{(d-4)(d-2)^2} \left(\tilde{R}^{ij} \tilde{R}_{ij} - \frac{d}{4(d-1)} \tilde{R}^2 \right) + \dots \right] \tag{188}$$

where $\tilde{g}$ is the induced metric on the brane and L is the scale of AdS_d. The total action of the brane is thus $S = I_{di} + I_{brane}$. The brane action is usually simply given by

$$I_{brane} = -T \int d^d x \sqrt{-\tilde{g}}, \tag{189}$$

like the one we used in the BCFT[5]. To illustrate this procedure, let us work out the detail for case of $d = 2$. Let us formulate our set-up first. We will study a holographic system, where the boundary theory is a 2D CFT which couples to a codimension-1 conformal defect. The bulk description of the system is an AdS_3 with a codimension-one brane. The AdS_3 metric is given by (184):

$$ds^2 = d\rho^2 + \cosh^2(\rho/L) g_{ij}^{AdS_2} dx^i dx^j, \tag{190}$$

$$g_{ij}^{AdS_2} dx^i dx^j = L^2 \left[-\cosh^2 r dt^2 + dr^2 \right]. \tag{191}$$

Then we replace the ρ coordinate with a Fefferman-Graham coordinate

$$z = 2L e^{-\rho/L}, \tag{192}$$

which leads to

$$ds^2 = \frac{L^2}{z^2} \left[dz^2 + \left(1 + \frac{z^2}{4L^2} \right)^2 g_{ij}^{AdS_2} dx^i dx^j \right]. \tag{193}$$

The asymptotic boundary is at $z = 0$ and $z = \infty$. Assume that the brane cuts off the AdS_3 geometry at $z = z_B$. Given this metric one can compute the Ricci scalar and extrinsic curvature

$$R = -\frac{6}{L^2}, \tag{194}$$

$$K_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial n} |_{z_B} = \frac{z}{2L} \frac{\partial g_{ij}}{\partial z} |_{z_B}, \quad K = -2 \left(\frac{L}{z_B^2} - \frac{z_B^2}{16L^3} \right) \tag{195}$$

[5] while we can also choose something else

where $g_{ij} = \frac{L^2}{z^2}(1+\frac{z^2}{4L^2})^2 g_{ij}^{AdS_2}$ and $K = K_{ij}g^{ij\,AdS_2}$. Therefore the gravitational action on the brane is given by

$$\begin{aligned} I_{di} &= \frac{1}{16\pi G_3}\int d^2x\sqrt{-g^{AdS_2}}\left(\int_{z_B}^{\infty} dz\frac{(4L^2+z^2)^2}{16Lz^3}(-\frac{6}{L^2}+\frac{2}{L^2}) - 4(\frac{L}{z_B^2}-\frac{z_B^2}{16L^3})\right) \\ &= \frac{1}{16\pi G_3}\int d^2x\sqrt{-g^{AdS_2}}\left(\frac{2L}{z_B^2}+\frac{2}{L}\log\frac{z_B}{L}-\frac{z_B^2}{8L}+\dots\right) \end{aligned} \tag{196}$$

where we have dropped the contribution from the infinity. Next we rewrite the expression in terms of the induced metric via the relations

$$\sqrt{-\tilde{g}} = \frac{L^2}{z_B^2}(1+\frac{z_B^2}{4L^2})^2\sqrt{-g^{AdS_2}}, \quad \tilde{R} = -2\frac{z_B^2}{L^4}(1+\frac{z_B^2}{4L^2})^{-2}. \tag{197}$$

The result is

$$I_{di} = \frac{L}{16\pi G_3}\int d^2x\sqrt{-\tilde{g}}\left[\frac{2}{L^2}+\frac{1}{2}\tilde{R}+\frac{L^2}{16}\tilde{R}^2-\frac{1}{2}\tilde{R}\log(-\frac{L^2\tilde{R}}{2})\right]. \tag{198}$$

The first two terms exactly give rise to the Hilbert-Einstein action on the 2D brane and the third term can be understood the 1-loop correction. The last term involved with logarithm is related to the conformal anomaly.

8.2 Doubly holographic model

In this doubly holographic model we will choose the brane action to be

$$I_{brane} = I_{JT} + I_{ct}, \tag{199}$$

where I_{ct} is included to cancel the first term in (198). Therefore the full induced the action (including the coupled CFT action) is

$$S = I_{di} + I_{brane} = \frac{1}{16\pi G_2}\int d^2x\sqrt{-\tilde{g}}\left[\tilde{\Phi}_0\tilde{R} + \Phi(\tilde{R}+2)\right] + \dots + S_{CFT}(\tilde{g},\chi), \tag{200}$$

with the constant value shifted as

$$\tilde{\Phi}_0 = \Phi_0 + \frac{G_2}{G_3}, \tag{201}$$

where we have ignored the higher order R terms. With the RS model, this doubly holographic model can be straightforwardly generalized to higher dimensional cases. In [13], the brane is very close to the boundary of AdS_3:

$$g_{ij}^{(3)}|_{bdy} = \frac{1}{\epsilon^2}g_{ij}^{(2)}, \quad g_{ij}^{(2)} \equiv \tilde{g}, \tag{202}$$

and the brane is also called the Plank brane.

Next we couple this system to an external bath which is described by the same CFT_2 living on a non-gravitational flat spacetime.

The combined system has three alternative descriptions:

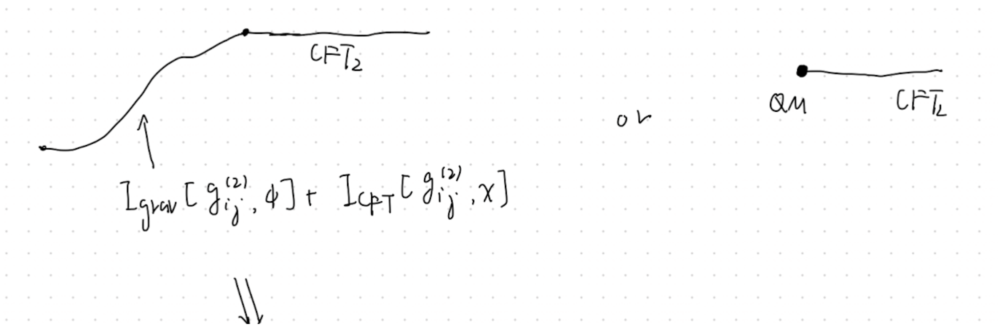

Figure 13: **Doubly holographic model**

1. **2d-Gravity**: A two-dimensional gravity-plus-matter theory living on $\sigma_y < 0$ coupled to a two-dimensional field theory living on $\sigma_y > 0$.
2. **3d-Gravity**: A three-dimensional gravity theory in AdS_3 with a dynamical boundary (Plank brane) on part of the space ($\sigma_y < 0$), and with a rigid boundary on the rest ($\sigma_y > 0$).
3. **QM**: A two-dimensional CFT on the half-line $\sigma_y > 0$ with some non-conformal boundary degrees of freedom at $\sigma_y = 0$.

In the third description, we have assumed that the 2D system has $(0+1)$ quantum-mechanical dual. Then the holographic derivation (in the second description) of Page curve is the following (Here we follow T. Takayanagi's description):

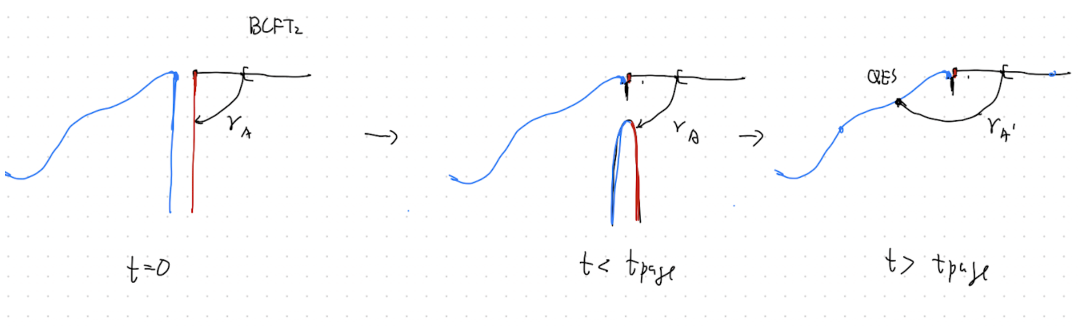

Figure 14: **Page curve**

1. At $t = 0$, the bath CFT and the 2D gravitational system are disconnected so the RT surface of interval A in the bath is simple Γ_A which ends on the boundary.
2. Before the Page time, even though the two system start to connect, the dominate RT surface of interval A in the bath is still Γ_A which ends on the boundary.

3. After the Page time, the dominate RT surface is the one ending on the Plank brane. So there is a phase transition.

We finish the section with some comments:

1. It is kind of artificial to to choose $I_{brane} = I_{JT} + I_{ct}$, it would be nice to derive the double holographic model more natually from RS model.
2. We should notice that the BCFT holographic entanglement entropy can explain part of the island formula. Even though we argue that the missing piece can be added back by considering brane world. But this is still a proposal not a proof. We just replace the island proposal with the a BCFT+braneworld proposal.
3. We have shown that the island exists in the simple 2D model. Higher dimensional islands are also found numerically for example in [20].
4. In our simple 2D double sided black hole example, the new QES is outside the horizon. But if decouple the black hole from the bath, the decoupling process will inevitably produce energy flux into the black hole so that the horizon will move outward. The QES then will lie behind the horizon as the situation in the one-sided black hole model [12]. In general, QES should be behind the horizon due to quantum focusing conjecture [21].
5. We only derive the Page curve with island formula in the AdS+non-gravitational bath system. Optimistically we expect that the Page curve of black hole in asymptotically flat spacetime can be derived in the same way: there is islands appearing at the Page time, for example [22][6]. The idea is straightforward: we can choose a cut-off surface which is way from the black hole horizon to separate the spacetime into two regions: the black hole region and the flat bath region. In the flat bath region gravity may be ignored. However this may be not correct. In [24], it was shown the non-gravitational bath is crucial. The bath is not just an auxiliary spectator but it actually influences the physics. If we make the bath gravitating, the Page curve will disappear. So it implies that we should not ignore the gravity in the asymptotically flat region just because the gravity is weak far away from the black hole. At least it needs more careful justification.

9 Baby Universe

We have derived the Page curve with the island formula but have we resolved information paradox? For example, we can ask ourselves the following questions:

1. Is Hawking radiation thermal or pure? Both Hawking's calculation and the island formula only use semi-classical approximation why the results are different? The island formula is just a trick to produce Page curve or it reflects real life physics?

[6]but there seems also a counterexample [23]

2. If we can collect all the Hawking radiation of an evaporated black hole how to know the state of Hawking radiation by doing local measurements?
3. What is the difference between replica wormholes and spacetime wormholes.
4. ...

If we believe black hole unitarity of course Hawking radiation is pure. But currently we do not know how to compute the exact density matrix of the Hawking radiation in a quantum gravity theory. Hawking's semi-classical calculation only gives some coarse grained von Neumann entropy while by taking into account new saddles the island formula gives the fine grained entropy which does follow the Page curve. These new saddles can be understood as replica wormholes. The physics we learn from island formula or replica wormhole is that in the Euclidean path integral we also need to consider saddles which correspond to (complex) singular geometry because even though these semi-classical solutions are singular but they give finite contribution to the action just like other solitons or instantons. Have some asymptotic observer collect all the Hawking radiation of a black hole, we only have one copy of the density matrix ρ but no measurement on a single copy can help us to distinguish a mixed state from an unknown pure state. Therefore the asymptotic observer needs form and evaporate a large number n copies of black holes which are largely separately in spacetime, and make joint measurements on the resulting n copies of Hawking radiation. This set-up may be interpreted as a physical version of the replica trick. The crucial difference is that now each replica is a physical system.

Hawking's prediction of the density matrix is $\rho_{Hawking}$ which is thermal. To prove or disprove this prediction the experimenter can perform a swap test. For a simple example of $n = 2$, the acts of the swap operator $\mathcal{S}$ is to exchange two replicas: $\mathcal{S}|\psi_1\rangle \otimes \psi_2\rangle = |\psi_2\rangle \otimes \psi_1\rangle$ such that expectation value of this operator is

$$\langle \mathcal{S} \rangle = \mathrm{Tr}(\mathcal{S}\rho \otimes \rho) = \mathrm{Tr}(\rho^2), \tag{203}$$

which is as known as the purity of ρ. More generally, on n replica we can measure the expectation value of the cyclic permutation operator U_τ: $\mathrm{Tr}(U_\tau \rho^{(n)})$ or the so-called swap entropies

$$S_n^{swap}(\rho^{(n)}) \equiv -\frac{1}{n-1} \log \mathrm{Tr}(U_\tau \rho^{(n)}), \tag{204}$$

which is the physical quantity the experimenter can measure. So the experimenter should ask Hawking to provide his prediction of (204) instead of naively considering $S_n^{swap}(\rho_{Hawking}^{\otimes n})$. Hawking's theoretical prediction will be a path integral calculation of (204). To compute (204) even in the semi-classical level is very complicated because we need to find all the possible saddles. Our approach is to focus on the saddles which we have known: Hawking saddles, Polchinski-Strominger saddles and replica wormhole saddles. The main reference is [25][7].

[7]For an overview of [25], see [26].

9.1 Hawking saddles

This is the set-up of Hawking' calculation of $\rho_{Hawking}$. We will use a in-in formalism to compute the density matrix so we need two copies of black hole geometry for ket and bra states, see Fig.(9.1)

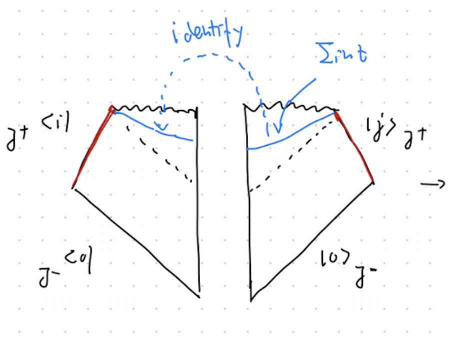

Figure 15: **Hawking saddle I**

First we perform a Euclidean path integral over the half-infinite flat space time to prepare the vacuum state at $\mathscr{I}_-$ then perform a Lorentzian path integral forward with boundary conditions at $\mathscr{I}_+$ are $\langle i|$ and $|j\rangle$. The internal surfaces Σ_{int} are identified to denote a trace over the states in the black hole. The result is the density matrix ρ_{ij}. Similarly we can compute the density matrix ρ_u associated with partial Hawking radiation as shown in Fig.(9.1)

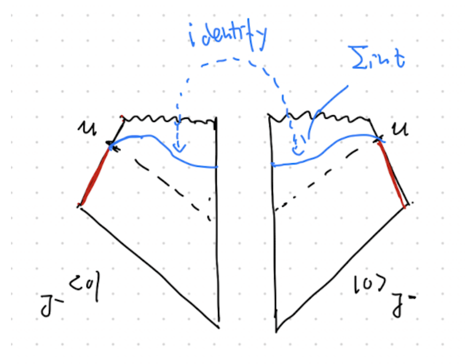

Figure 16: **Hawking saddle II**

9.2 Polchinski-Strominger saddles

Let us consider the geometry of the whole evaporation process shown in Fig[INSERT FIGURE] and assume

1. In the full geometry Fig. (1),

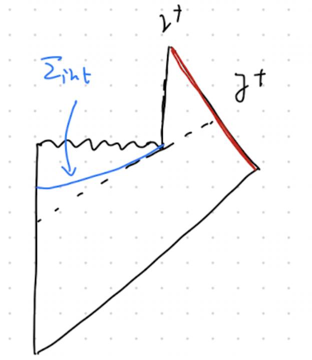

Figure 17: **PS saddle**

   the spacetime is empty near future timelike infinity i^+.

2. For any Cauchy surface Σ_{int} of the black hole interior, we may treat $\mathscr{J}^+ \cup \Sigma_{int}$ as a (disconnected) Cauchy surface for the full spacetime.

We have stressed that Σ_{int} and $\mathscr{J}^+$ are disconnected so after the black hole evaporation this part Σ_{int} is detached from the original universe and we will think Σ_{int} belongs so-called baby universe. Moreover there is no way to distinguish different Σ_{int} from different copies in computing $\rho^{(n)}$ so they should be indistinguishable and satisfy the Bose statistics. Therefore are $n!$ possible saddles which are related by permutations and components of the density matrix is given by

$$\langle i_1, \dots, i_n | \rho^{(n)} | j_1, \dots, j_n \rangle = \sum_{\pi \in \mathrm{Sym}(n)} \langle i_1 | \rho_{Hawking} | j_{\pi(1)} \rangle \dots \langle i_n | \rho_{Hawking} | j_{\pi(n)} \rangle . \tag{205}$$

The Fig.(9.2)

shows the two saddles for $n = 2$ case. Taking the new saddles into account the purity is given by

$$\mathrm{Tr}(\mathcal{S}(\mathscr{J}_u)\rho^{(2)}) = \mathrm{Tr}(\mathcal{S}(\mathscr{J}_u)\rho_{Hawking}^{\otimes 2}) + \mathrm{Tr}(\mathcal{S}(\mathscr{J}_u)\mathcal{S}(\mathscr{J}^+)\rho_{Hawking}^{\otimes 2}). \tag{206}$$

Since $\mathscr{J}_u + \mathscr{J}_{\bar{u}} = \mathscr{J}^+$ then $\mathcal{S}(\mathscr{J}_u)\mathcal{S}(\mathscr{J}^+) = \mathcal{S}(\overline{\mathscr{J}_u})$. Thus the purity is equal to

$$S_2^{swap}(u) \sim \min\{S_2^{Hawking}, \bar{S}_2^{Hawking}\}, \tag{207}$$

where we have approximate the function as a minimum of the two terms because we expect one of them is large. So the path integral calculation will produce the Page curve! It implies that the n copies of density matrix are not uncorrelated as we expect. The correlation is mediated through the baby universe. Alternatively we can think of that different copies of black holes are actually connected by spacetime wormholes. Of course considering wormholes will violate the cluster decomposition and it also causes other problems. For example, as we shown in the JT gravity case, it will change the inner product between states. The Polchinski-Strominger baby universe proposal can not be accepted mainly due to following challenges:

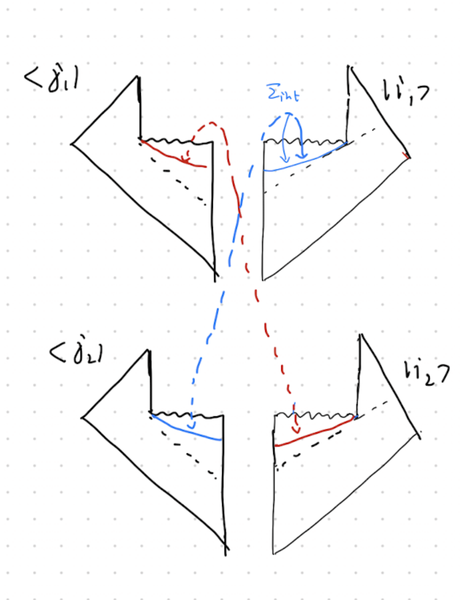

Figure 18: **PS saddle**

1. The PS saddle geometries include the end point of the evaporation (which is a point on Σ_{int}) where we lose the semi-classical control.
2. The Bekenstein-Hawking entropy S_{BH} bound is violated. Because the $S_2^{Hawking} + \bar{S}_2^{Hawking} = S^{Hawking}(\infty)$ which exceeds the bound and

$$\frac{\bar{S}_2^{Hawking}(u)}{S_{BH}(u)} = \frac{S^{Hawking}(\infty)}{S_{BH}(0)} > 1 \tag{208}$$

3. It violates the causality. Because the quantity $\bar{S}_2^{Hawking}(u)$ depends on the entire future of the black hole but some how we can perform the swap experiment to obtain it at time u. (Here we have assumed that when we do the swap test, the measurement will not change the semi-classical geometry. Basically we do not consider the Schrodinger's cat scenerio.)

9.3 Replica saddles

We can think of that replica saddles are upgraded Polchinski-Strominger saddles. The key point is that we choose a new Cauchy surface $\mathscr{I}_u \cup \Sigma_{ext} \cup \mathcal{I}$ instead of the disconnected $\mathscr{I}^+ \cup \Sigma_{int}$ so that on this new Cauchy surface semi-classical description is always applicable. And the baby universe region Σ_{int} is replaced by the island $\mathcal{I}$. $\mathcal{I}$ and Σ_{ext} meets at a codimension-2 boundary $\partial I = \gamma$ which can be thought of as a gate to the baby universe. Different copies of ∂I can be sewn together along γ. In the sewn geometry, there will a conical singularity at γ so the γ codimension-2 boundary will be the QES as we expect. The Fig. (9.3)

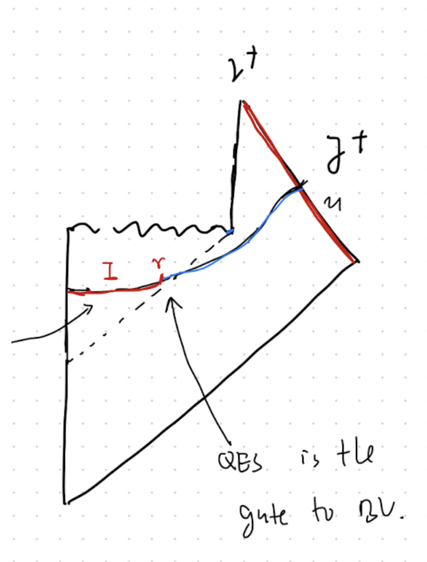

Figure 19: **Replica saddle**

shows the two saddles for $n = 2$ case. With the new saddles the purity becomes

$$S_2^{swap}(u) \sim \min\{S_2^{Hawking}, S_{BH}\}, \tag{209}$$

where we have used the fact γ is very close the horizon.

9.4 Hilbert space of baby universes and ensembles

We have shown how the replica wormholes and baby universes can introduce correlations between replicas from the path integral perspective. Now let us describe the Hilbert space interpretation of the correlations. We will focus on the case of PS wormhole since the generalization to the case of replica wormhole is straightforward. A single PS ket spacetime with boundary conditions imposed on $\mathscr{I}^+$ and Σ_{int} computes the wavefunction ψ_{ai} of a state in $\mathcal{H}_{\mathscr{I}^+} \otimes \mathcal{H}_{int}$:

$$|\psi\rangle = \sum_{i,a} \psi_{ai} |i\rangle_{\mathscr{I}^+} \otimes |a\rangle_{int}. \tag{210}$$

Identifying the Σ_{int} of the ket spacetime and bra spacetime gives the Hawking density matrix (one may find this is exactly the same calculation as we did in section (7))

$$\rho_{Hawking} = \sum_{i,j,a,b} \bar{\psi}_{bj} \psi_{ai} \langle b|a\rangle_{int} (|i\rangle\langle j|)_{\mathscr{I}^+}. \tag{211}$$

If we choose an orthonormal basis $\langle b|a\rangle_{int} = \delta_{ab}$ then we have

$$(\rho_{Hawking})_{ij} = \sum_a \bar{\psi}_{aj} \psi_{ai} \equiv (\psi_j, \psi_i). \tag{212}$$

Now define the Hilbert space the baby universe as

$$\oplus_{n=0}^{\infty}\mathrm{Sym}^n\mathcal{H}_{int} \equiv \mathcal{H}_{BU}. \tag{213}$$

Then the elements of the density matrix of $\rho^{(n)}$ can be written as

$$\langle i_1,\ldots,i_n|\rho^{(n)}|j_1,\ldots,j_n\rangle = \sum_{a,b}\psi_{a_1 i_1}\bar{\psi}_{b_1 j_1}\ldots\psi_{a_n i_n}\bar{\psi}_{b_n j_n}\langle b_1,\ldots,b_n|a_1,\ldots,a_n\rangle_{BU}. \tag{214}$$

with

$$\langle b_1,\ldots,b_n|a_1,\ldots,a_n\rangle_{BU} = \sum_{\pi\in\mathrm{Sym}}\delta_{a_1 b_{\pi(1)}}\ldots\delta_{a_n b_{\pi(n)}}. \tag{215}$$

The definition (213) can be understood as the Fock space then we can associate the states $|a\rangle$ and $\langle b|$ with creation and annihilation operators

$$|a_1,\ldots,a_n,\bar{b}_1,\ldots,\bar{b}_n\rangle = A^\dagger_{a_1}\ldots A^\dagger_{a_n}B^\dagger_{b_1}\ldots B^\dagger_{b_n}|HH\rangle, \tag{216}$$

$$[A_a,A^\dagger_a] = [B_a,B^\dagger_a] = \delta_{ab}, \quad A_a|HH\rangle = B_a|HH\rangle = 0, \tag{217}$$

where $A_a(B_a)$ and $A^\dagger_a(B^\dagger_a)$ annihilate and create a (anti) baby universe in the state $a(b)$ and $|HH\rangle$ is the vacuum or the zero-universe state.

Alternatively we can understand the baby universe states in the following way. As before we think of $|HH\rangle$ to denote a closed surface state (or the no boundary state). The boundary operators can be defined as

$$\hat{\alpha}_a = A^\dagger_a + B_a, \tag{218}$$

thus

$$\langle b_1,\ldots,b_n|a_1,\ldots,a_n\rangle_{BU} = \langle\alpha_{a_1},\ldots\alpha_{a_n}\bar{\alpha}_{b_1}\ldots\bar{\alpha}_{b_n}\rangle_{BU} \tag{219}$$

$$\langle\alpha_{a_1},\ldots\alpha_{a_n}\bar{\alpha}_{b_1}\ldots\bar{\alpha}_{b_n}\rangle_{BU} = \langle HH|\alpha^\dagger_{b_1}\ldots\alpha_{a_n}|HH\rangle \tag{220}$$

The first identity (219) means that the inner product can be understood as an ensemble average of random variables α and $\bar{\alpha}$ satisfying the Bose statistics

$$\langle F[\alpha,\bar{\alpha}]\rangle_{BU} = \int\prod_a d\alpha_a d\bar{\alpha}_a e^{-\sum_a\alpha_a\bar{\alpha}_a}F[\alpha,\bar{\alpha}]. \tag{221}$$

The second identity (220) means that the ensemble average can also be understood as correlation functions of a quantum gravity theory. With this representation, the elements of the density matrix can be written as

$$\langle i_1,\ldots,i_n|\rho^{(n)}|j_1,\ldots,j_n\rangle = \langle\bar{\Psi}_{j_1}\ldots\bar{\Psi}_{j_n}\Psi_{i_1}\ldots\Psi_{i_n}\rangle_{BU}, \tag{222}$$

$$\Psi_i = \sum_a\alpha_a\psi_{ai}. \tag{223}$$

Thus $\hat{\Psi}_i = \sum_a\hat{\alpha}_a\psi_{ai}$ can also be treated as a boundary-inserting operator which is specified by boundary conditions of quantum gravity path integral. Since the order of boundary condition is relevant for the path integral, these operators commute to each other i.e. $[\alpha, \alpha^\dagger] = [\alpha, \alpha] = [\alpha^\dagger, \alpha^\dagger] = 0$. This is also the main reason we introduce B and $B^\dagger$ in our definition even though they do not appear in our discussion (the inner product is always involved with baby universe not anti- baby universe states.). As a consequence, they can be diagonalized at the same time and the common eigenfunction is usually called the α–state (superselection sector). Fixing each α_a to some value in (222) or equivalently replacing $|HH\rangle$ with some α–state, the density matrix will factorize

$$\langle i_1, \ldots, i_n | \rho^{(n)} | j_1, \ldots, j_n \rangle \quad \to_{\text{fix } \alpha} \quad \Psi^\alpha_{i_1} \bar{\Psi}^\alpha_{j_1} \ldots \Psi^\alpha_{i_n} \bar{\Psi}^\alpha_{j_n} \tag{224}$$

which implies the Hawking radiation $|\Psi^\alpha\rangle \in \mathcal{H}_{\mathscr{I}^+}$ is a pure state. As a result, we can think of the Hawking radiation is in a superposition state of different superselection sector but our theory can not give a specific prediction for $|\Psi^\alpha\rangle$. Instead, it only gives a result after a probabilistic average for example:

$$\rho_{Hawking} = \int d\mu(\alpha) |\Psi^\alpha\rangle\langle\Psi^\alpha|. \tag{225}$$

In the end, let us go back to the JT gravity example. The full state is (164)

$$|\Psi\rangle = \sum_i |i\rangle \otimes |\psi_i\rangle = \sum_{i,a} \psi_i^a |i\rangle \otimes |a\rangle \tag{226}$$

with the inner product (175)

$$\langle \psi_i | \psi_j \rangle = \delta_{ij} + \sum_n \lambda^n R_{ij}^{(n)}, \quad \lambda = e^{-S_0}. \tag{227}$$

Assume that we can diagonalize the inner product and find the new orthonormal basis $|a\rangle = \sum_i \psi_i^a |\psi_i\rangle$. Then we can define $\hat{\alpha}$ operator as before

$$\hat{\alpha}_a = (A_a^\dagger + B_a) = \sum_i (\psi_i^a \hat{\psi}_i^\dagger + \psi_i^a \hat{\bar{\psi}}_i), \tag{228}$$

where $\hat{\bar{\psi}}_i$ corresponds to the anti operator. Therefore we can identity the boundary operator of JT gravity as

$$(\psi_i, \psi_j) = (\hat{\psi}_i + \hat{\bar{\psi}}_i^\dagger)(\hat{\psi}_j^\dagger + \hat{\bar{\psi}}_j) = \sum_{a,b} (\bar{\psi}_i^a)^{-1} (\psi_j^b)^{-1} \hat{\alpha}_a^\dagger \hat{\alpha}_b. \tag{229}$$

A SYK model

This section is a review of SYK model. Hopefully the review can explain the relation between SYK and JT gravity. The other goal of this section is to understand some novel properties of SYK model including the solvability, emergent conformal symmetry and reparameterization invariance and the nature of being an ensemble theory. All these properties make SYK to be an interesting toy model for studying strong coupling systems with many degrees of freedom.

The model

SYK model is quantum mechanics of $N = 2K \gg 1$ Majorana fermions with all-to-all couplings:

$$I_{SYK} = \int d\tau \left[\frac{1}{2} \sum_{i=1}^{N} \chi_i(\tau)\dot{\chi}_i(\tau) - \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\chi_i(\tau)\chi_j(\tau)\chi_k(\tau)\chi_l(\tau) \right], \quad (230)$$

where τ is the Euclidean time and χ_i are Hermitian operators and obey the anti-commutation relations:

$$\{\chi_i, \chi_j\} = \delta_{ij}, \quad i, j = 1, \ldots, N. \quad (231)$$

To parameterize this algebra we can think that each χ_i is a $2^K \times 2^K$ matrix. Via the Legendre transformation the Hamiltonian is

$$H = \frac{1}{4!} \sum_{i,j,k,l=1}^{N} J_{ijkl}\, \chi_i(\tau)\chi_j(\tau)\chi_k(\tau)\chi_l(\tau), \quad (232)$$

which is also a $2^K \times 2^K$ matrix. The coupling J_{ijkl} are not constants but follow the Gaussian distribution with mean and variance

$$\langle J_{ijkl} \rangle = 0, \quad \langle J_{ijkl}^2 \rangle = \frac{3! J^2}{N^3}. \quad (233)$$

By diagonalizing this matrix one can find the spectrum of the model. The coupling J_{ijkl} is relevant so the theory is expected to asymptotically free at very large energy. In this large energy limit, the Hamiltonian is simple vanishing. The propagator or the two-point function is

$$G_{ij}^f = \langle T\chi(\tau)\chi(0)\rangle = \langle \chi_i(\tau)\chi_j(0)\rangle\theta(\tau) - \langle \chi_j(0)\chi_i(\tau)\rangle\theta(-\tau) = \frac{1}{2}\delta_{ij}\mathrm{sgn}\tau. \quad (234)$$

We can also derive this propagator from the standard rule

$$G_{ij}^f = \delta_{ij}\frac{1}{\partial_\tau}, \quad \rightarrow \quad G^f(\omega) = -\frac{1}{i\omega}. \quad (235)$$

Higher point correlation functions are given by Wick contraction. Use this free propagator we can then compute corrections due to the interaction perturbatively

$$\begin{aligned} G(\tau) \quad &= \quad \langle T[\chi_n(\tau)\chi_m(0) + \frac{1}{4!}\sum_{ijkl} J_{ijkl} \int dt' \chi_n(\tau)\chi_m(0)\chi_i'\chi_j'\chi_k'\chi_l' \qquad (236) \\ &+ \quad \frac{1}{2}\frac{1}{(4!)^2} \sum_{ijkl,pqrs} J_{ijkl}J_{pqrs} \int dt' \int dt'' \chi_n(\tau)\chi_m(0)\chi_i'\chi_j'\chi_k'\chi_l'\chi_p''\chi_q''\chi_r''\chi_s'' + \mathcal{O}(J^3)]\rangle. \end{aligned}$$

Doing the ensemble average by the Wick contraction rule

$$\langle J_{ijkl}J_{pqrs}\rangle = \frac{3!J^2}{N^3}\sum_{\sigma}\mathrm{sgn}(\sigma)\delta_{i\sigma p}\delta_{j\sigma q}\delta_{k\sigma r}\delta_{l\sigma s}. \tag{237}$$

in the large N limit, one can find that the leading contribution comes from the contraction

$$\sum_{klm}\langle J_{iklm}J_{jklm}\rangle = 3!J^2\delta_{ij} + \mathcal{O}(\frac{1}{N}), \tag{238}$$

which correspond to the melon diagrams. Then the propagator in the large N limit is given by the summation of the geometric series

$$\begin{aligned} G &= G^f + G^f\Sigma G^f + G^f\Sigma G^f\Sigma G^f + \ldots \\ &= [(G^f)^{-1} - \Sigma]^{-1} = [\partial_\tau - \Sigma]^{-1}, \quad \Sigma = J^2G^3. \end{aligned} \tag{239}$$

The Fourier transformation (where we used the translation symmetry) of the first equation is

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega). \tag{240}$$

These equations (239) and (240) are also known as the Dyson-Schwinger equation and they can be solved numerically by iterations so in this sense the SYK model is solvable at large N. This is the same solvability of the vector model where the only leading diagram is the bubble diagram.

There is another distinct solvability which is absent in the vector model. Let us consider the IR property, the same as the strong coupling (recall J relevant), of the solution. J is only scale of the model, so the IR limit means the frequencies $\omega \ll J$ (or for a thermal solution means $\beta J \gg 1$). In this limit, the term $i\omega$ drops. Then the DS equation is approximated as

$$\int d\tau' G(\tau,\tau')\Sigma(\tau',\tau'') = -\delta(\tau-\tau''), \quad \Sigma(\tau,\tau') = J^2|G(\tau,\tau')|^3. \tag{241}$$

They are invariant under reparametrizations $\tau \to \phi(\tau)$ if the fields transform as

$$G(\tau,\tau') \;\to\; [\phi'(\tau)\phi'(\tau')]^{1/4}G(\phi(\tau),\phi(\tau')), \tag{242}$$

$$\Sigma(\tau,\tau') \;\to\; [\phi'(\tau)\phi'(\tau')]^{3/4}\Sigma(\phi(\tau),\phi(\tau')). \tag{243}$$

They can be thought of as two primary point functions of a conformal field theory. So a possible solution is

$$G_c(\tau) = \frac{b}{|\tau|^{2\Delta}}\mathrm{sgn}(\tau), \quad \Delta = \frac{1}{4} \tag{244}$$

and the prefactor b can be fixed by substituting this ansatz to the equation. One can solve

$$b^4 = \frac{1}{\pi J^2}(\frac{1}{2} - \frac{1}{4}) \tan\frac{\pi}{4}. \tag{245}$$

Other solutions are obtained by reparametrizations of this solution:

$$G_c(\tau_1, \tau_2) = b\,\mathrm{sgn}(\tau)\frac{\phi'(\tau_1)^\Delta \phi'(\tau_2)^\Delta}{|\phi(\tau_1) - \phi(\tau_2)|^{2\Delta}}. \tag{246}$$

For example the two point function on the thermal circle $\tau \sim \tau + \beta$ is given by applying the transformation $\phi(\tau) = \tan\frac{\pi\tau}{\beta}$. The result is

$$G_c(\tau) = b(\frac{\pi}{\beta \sin\frac{\pi\tau}{\beta}})^{1/2}\mathrm{sgn}(\tau). \tag{247}$$

However there is a degeneracy in (246). When ϕ is a transformation in $SL(2,R)$, the Mobius transformation, then (246) still gives (244). So the space of solution is

$$\mathrm{Diff}S^1/SL(2,R). \tag{248}$$

Choosing one of them, say (244), the reparametrization invariance is spontaneously broken down to $SL(2,R)$. It implies[8] that at the IR fixed point all Goldstone modes of SYK can be described by a one-dimensional CFT. Let us get some understanding about these Goldstone modes from two examples. The first example is the asymptotic symmetry of AdS_3 space. After choosing the AdS_3 space, the asymptotic symmetry is broken down to the global conformal symmetry. The resulted Goldstone modes are described by the coadjoint orbit of the Virasoro group

$$\mathrm{Diff}S^1/SL(2,R) \oplus \mathrm{Diff}S^1/SL(2,R). \tag{249}$$

Other other example is the relativistic particle whose action is

$$S = \frac{1}{2}\int d\tau\, e(\tau)\left[e^{-2}(\tau)\dot{x}\cdot\dot{x} - m^2\right]. \tag{250}$$

The action is also invariant under representation of the worldline provided

$$e'(\tau')d\tau' = e(\tau)d\tau. \tag{251}$$

The propagator is given by the path integral

$$\langle x'|x\rangle = \int_{x(0)=x}^{x(t)=x'} \mathcal{D}e\mathcal{D}x \exp\left[-\frac{i}{2}\int_0^t (\frac{1}{e}\dot{x}^2 - e\,m^2)d\tau\right] \tag{252}$$

[8] This perspective is examined for example in [?].

Because of the reparameterization invariance this integral will be divergent. Here we should treat reparameterization invariance as a gauge symmetry of the relativistic particle do a gauge fixing. By the gauge fixing we can remove all the modes except for the zero modes,i.e.

$$L = \int_0^t d\tau e/t \quad \rightarrow \quad e = L. \tag{253}$$

Then the gauge-fixed path integral is

$$\langle x'|x\rangle = \hat{N} \int_0^\infty dL \int_{x(0)=x}^{x(1)=x'} \mathcal{D}x \exp\left[-\frac{1}{2}\int_0^1 (\frac{1}{L}\dot{x}^2 - L\, m^2) d\tau\right], \tag{254}$$

where we have added the normalization factor, rescaled $t = 1$ and rotate to the Euclidean time $\tau \rightarrow -i\tau$. To evaluate this path integral we can expand it around the classical path

$$x(\tau) = x + (x' - x)\tau + \delta x. \tag{255}$$

The measure for the fluctuations is

$$||\delta x||^2 = \int_0^1 d\tau e(\delta x)^2 = L \int_0^1 d\tau (\delta x)^2 \tag{256}$$

such that

$$Dx \sim \prod_\tau \sqrt{L} d\delta x(\tau). \tag{257}$$

Then we arrived at the final expression

$$\begin{aligned} \langle x|x'\rangle &= \hat{N} \int_0^\infty dL \int \prod \sqrt{L} d\delta x(\tau) e^{-(x'-x)^2/2L - m^2 L/2} e^{-(1/2L)\int_0^1 d\tau (\delta\dot{x})^2} \\ &= \hat{N} \int_0^\infty dL\, e^{-(x'-x)^2/2L - m^2 L/2} [\det(-\frac{\partial_\tau^2}{L^2})]^{-D/2} \\ &= \hat{N'} \int_0^\infty dL\, e^{-(x'-x)^2/2L - m^2 L/2} L^{-D/2}. \end{aligned} \tag{258}$$

In the second line we have absorbed the term which is divergent and needs regularization into the normalization since it is a only a constant factor. The $L^{-D/2}$ comes from

$$\det(1/L^2) = \prod L^{-2\zeta(0)} = L. \tag{259}$$

We may try to perform a similar calculation for the SYK model in the low energy conformal limit. First we observe that these equations of motion (241) can de derived from the effective action

$$S_c^{eff} = -\frac{1}{2} \int \int \log(-\Sigma)\delta(\tau - \tau') + \frac{1}{2} \int \int (\Sigma G - \frac{1}{4} G^4). \tag{260}$$

Similarly, because of the reparameterization invariance the path integral will be divergent. Let us see this divergence more carefully from another interesting quantity the four point function of fermions:

$$\frac{1}{N^2}\sum_{ij}\langle\chi_i(\tau_1)\chi_j(\tau_2)\chi_k(\tau_3)\chi_l(\tau_4)\rangle = \int d\Sigma dG e^{-S_c^{eff}} G(\tau_1,\tau_2)G(\tau_3,\tau_4). \tag{261}$$

Let us analyze the leading $1/N$ piece of the left-hand side:

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\langle\chi_i(\tau_1)\chi_j(\tau_2)\chi_k(\tau_3)\chi_l(\tau_4)\rangle = G(\tau_{12})G(\tau_{34}) + \frac{1}{N}\mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4) + \ldots \tag{262}$$

the first term is the disconnected piece. To compute $\mathcal{F}$, the diagrams which are needed to summed are the ladder diagrams with any number of rungs. The first diagram, $\mathcal{F}_0$, is just a product of propagotors

$$\mathcal{F}_0(\tau_1\ldots\tau_4) = -G_{\tau_{12}}G_{\tau_{24}} + G_{\tau_{14}}G_{\tau_{23}}. \tag{263}$$

The next diagram is the one-rung ladder:

$$\mathcal{F}_1 = 3J^2\int d\tau d\tau'\left[G(\tau_1-\tau)G(\tau_2-\tau')G(\tau-\tau')^2G(\tau-\tau_3)G(\tau'-\tau_4) - (\tau_3\leftrightarrow\tau_4)\right]. \tag{264}$$

The standard technique for computing the ladder diagram is to use the diagram building kernal to compute them recursively

$$\mathcal{F}_{n+1}(\tau_1,\tau_2,\tau_3,\tau_4) = \int d\tau d\tau' K(\tau_1,\tau_2;\tau_3,\tau_4)\mathcal{F}_n(\tau,\tau',\tau_3,\tau_4), \tag{265}$$

where the kernel is

$$K(\tau_1,\tau_2;\tau_3,\tau_4) = -3J^2G_{\tau_{13}}G_{\tau_{24}}G^2_{\tau_{34}}. \tag{266}$$

The sum of all ladder diagrams is then a geometric series

$$\mathcal{F} = \sum_{n=0}^{\infty}K^n\mathcal{F}_0 = \frac{1}{1-K}\mathcal{F}_0. \tag{267}$$

Therefore we need to diagonalize K. This is doable due to that K commutes with the conformal symmetry, so the its eigenfunctions are the conformal blocks. Here we summarize the procedures and results:

1. $\mathcal{F}$ and $\mathcal{F}_n$ are only functions of the cross ratio $u = \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}$.
2. The eigenfunctions are particular hypergeometric functions $\Psi_h(u)$ related to the conformal blocks of weight h.
3. The complete set of h are $h = \frac{1}{2} + is$ and $h = 2, 4, 6, 8, \ldots$

4. The four point function then is given by

$$\mathcal{F}(u) = \sum_h \Psi_h(u) \frac{1}{1-k_c(h)} \frac{\langle \Psi_h, \mathcal{F}_0 \rangle}{\langle \Psi_h, \Psi_h \rangle}, \tag{268}$$

   where $k_c(h)$ are the eigenvalues of the kernel K as a function of h. In particular $k_c(2) = 1$, which leads to the divergence which we expected. So $h = 2$ mode is the reparameterization mode and we expect when the model is moved away from the IR fixed point, the leading non-conformal contribution is determined by the first order shift in the $h = 2$ eigenvalues of the kernel.

To study the leading non-conformal contribution let us restore $i\omega$ term and look at the full effective action

$$\frac{S^{eff}}{N} = -\frac{1}{2} \int \int \log(-\partial_\tau - \Sigma)\delta(\tau - \tau') + \frac{1}{2} \int \int (\Sigma G - \frac{1}{4} G^4), \tag{269}$$

which can separated into the conformally-invariant and non-invariant parts $S_{CFT} + S_S$

$$\begin{aligned} \frac{S_{CFT}}{N} &= -\frac{1}{2} \log\det(-\Sigma) + \frac{1}{2} \int \int (\Sigma G - \frac{1}{4} G^4), \\ \frac{S_s}{N} &= -\frac{1}{2} \int \int G_f^{-1} G(\tau, \tau') \delta(\tau - \tau'), \quad G_f^{-1} = \partial_\tau. \end{aligned} \tag{270}$$

after a shift $\Sigma \to \Sigma - G_f^{-1}\delta(\tau - \tau')$. The non-conformal term S_s may be thought of as a “boundary” term which breaks the reparameterization invariance explicitly. [Probably there is a better way to derive Schwarzian.] To characterize the breaking, recall the entire space of the conformal solutions are

$$G(\tau_1, \tau_2) = b \,\mathrm{sgn}(\tau) \frac{\phi'(\tau_1)^\Delta \phi'(\tau_2)^\Delta}{|\phi(\tau_1) - \phi(\tau_2)|^{2\Delta}}. \tag{271}$$

So to characterize different symmetry breakings we need to specify the “boundary” behaviors

$$\lim_{\tau_1 \to \tau_2} G(\tau_1, \tau_2). \tag{272}$$

Here we make the simplest choice by the Tylor expanding with respect to $\tau_{12} = \tau_1 - \tau_2$ around the center point $\tau_+ = (\tau_1 + \tau_2)/2$:

$$G(\tau_1, \tau_2) \approx b \frac{\mathrm{sgn}(\tau_{12})}{|\tau_{12}|^{2\Delta}} (1 + \frac{\Delta}{6} \tau_{12}^2 \mathrm{Sch}(\phi(\tau_+)), \tau_+) \tag{273}$$

which leads to action

$$\frac{S_s}{N} = -\frac{C}{2J} \int d\tau \,\mathrm{Sch}[f(\tau), \tau]. \tag{274}$$

The constant coefficient C will be fixed numerally. The field $f(\tau)$ is referred to as the reparameterization mode or the soft mode and it is the Nambu-Goldstone mode.

A.1 $O(N)$ Vector model

The Lagrangian is

$$\mathcal{L} = \frac{i}{2}\bar{\psi}_i \partial \psi_i + \frac{1}{4} g (\bar{\psi}_i \psi_i)^2. \tag{275}$$

Introducing the auxiliary field $\sigma(x)$ the Lagrangian can be rewritten as

$$\mathcal{L} = \frac{i}{2}\bar{\psi}_i \partial \psi_i - \frac{1}{4g}\sigma^2 + \frac{1}{2}\sigma \bar{\psi}\psi. \tag{276}$$

The equation of motion of the auxiliary field gives

$$\sigma = g\bar{\psi}\psi. \tag{277}$$

Integrating out the fermions leads to the effective action

$$\frac{I_\sigma}{N} = \frac{1}{2}\log\det(i\partial + \sigma) - \frac{1}{4gN}\int \sigma^2. \tag{278}$$

Therefore in the large N limit, the first term dominates which corresponds to summing the one loop fermion diagrams at zero momentum.

B Thermofield double formalism

The thermofield double formalism is a trick to treat the thermal mixed state $\rho = e^{-\beta H}$ as a pure state in a bigger system. We consider a new QFT which is two copies of original QFT. The states in this doubled QFT are the tensor products of states of the two QFTs. The thermofield double state is defined as

$$|TFD\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\beta E_n}|n\rangle_1 |n\rangle_2. \tag{279}$$

The reduced density matrix of system 1 is then

$$\rho_1 = \mathrm{tr}_2 \rho_{TFD} = \sum_n e^{\beta E_n}|n\rangle \tag{280}$$

, so if we restrict system 1 we obtain a thermal state as we want. Now we see how to prepare this state by Euclidean path integral. We consider the a "cylinder" Σ:

$$\Sigma = \mathrm{Interval}_{\beta/2} \times S^{d-1}, \tag{281}$$

the interval is the length of the "cylinder". To confirm this state is really a thermofield double, let us compute the transition amplitude

$$\langle \varphi_1 | \langle \varphi_2 | TFD\rangle = \langle \varphi_1 | e^{-\beta H/2} | \varphi_2^\star\rangle = \sum_n e^{-\beta E_n/2}\langle \varphi_1 | n\rangle \langle \varphi_2 | n\rangle \tag{282}$$

up to some factor and reparametrization which is the desired matrix elements of the thermofield double state. Assume the QFT has a bulk gravity dual in the sense

$$Z_{gravity}[\partial M = \Sigma] = Z_{QFT}[\Sigma], \tag{283}$$

then we can prepare this thermofield double state by performing a path integral on M in the gravity theory. To do this we first need to find a Euclidean gravity solution with the boundary Σ. A obvious one is the half of the Euclidean black hole with Euclidean time range $t_E \in [0, \beta]$.

C Warped products

Consider the warped product geometry

$$\begin{aligned} ds^2 &= ds^2_{(x)} + e^{2w(x)} ds^2_{(y)}, \\ &= g_{\alpha\beta} dx^\alpha dx^\beta + e^{2w(x)} g_{mn} dy^m dy^n, \end{aligned} \tag{284}$$

and let k be the dimension of the y space The Ricci scalar factorizes as

$$R = R_x + e^{-2w} R_y - 2k \nabla^2_x w - k(k+1) g^{\alpha\beta} \partial_\alpha w \partial_\beta w. \tag{285}$$

References

[1] A. Almheiri and J. Polchinski, "Models of AdS$_2$ backreaction and holography," JHEP **11**, 014 (2015) doi:10.1007/JHEP11(2015)014 [arXiv:1402.6334 [hep-th]].

[2] J. Maldacena, D. Stanford and Z. Yang, "Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space," PTEP **2016**, no.12, 12C104 (2016) doi:10.1093/ptep/ptw124 [arXiv:1606.01857 [hep-th]].

[3] D. Harlow and D. Jafferis, "The Factorization Problem in Jackiw-Teitelboim Gravity," JHEP **02**, 177 (2020) doi:10.1007/JHEP02(2020)177 [arXiv:1804.01081 [hep-th]].

[4] D. Stanford and E. Witten, "Fermionic Localization of the Schwarzian Theory," JHEP **10**, 008 (2017) doi:10.1007/JHEP10(2017)008 [arXiv:1703.04612 [hep-th]].

[5] Z. Yang, "The Quantum Gravity Dynamics of Near Extremal Black Holes," JHEP **05**, 205 (2019) doi:10.1007/JHEP05(2019)205 [arXiv:1809.08647 [hep-th]].

[6] A. Blommaert, T. G. Mertens and H. Verschelde, "Fine Structure of Jackiw-Teitelboim Quantum Gravity," JHEP **09**, 066 (2019) doi:10.1007/JHEP09(2019)066 [arXiv:1812.00918 [hep-th]].

[7] P. Saad, S. H. Shenker and D. Stanford, "JT gravity as a matrix integral," [arXiv:1903.11115 [hep-th]].

[8] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, "The entropy of Hawking radiation," [arXiv:2006.06872 [hep-th]].

[9] A. Almheiri, R. Mahajan and J. Maldacena, "Islands outside the horizon," [arXiv:1910.11077 [hep-th]].

[10] G. Penington, S. H. Shenker, D. Stanford and Z. Yang, "Replica wormholes and the black hole interior," [arXiv:1911.11977 [hep-th]].

[11] P. Gao, D. L. Jafferis and D. K. Kolchmeyer, "An effective matrix model for dynamical end of the world branes in Jackiw-Teitelboim gravity," [arXiv:2104.01184 [hep-th]].

[12] G. Penington, "Entanglement Wedge Reconstruction and the Information Paradox," JHEP **09**, 002 (2020) doi:10.1007/JHEP09(2020)002 [arXiv:1905.08255 [hep-th]].

[13] A. Almheiri, R. Mahajan, J. Maldacena and Y. Zhao, "The Page curve of Hawking radiation from semiclassical geometry," JHEP **03**, 149 (2020) doi:10.1007/JHEP03(2020)149 [arXiv:1908.10996 [hep-th]].

[14] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, "The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole," JHEP **12**, 063 (2019) doi:10.1007/JHEP12(2019)063 [arXiv:1905.08762 [hep-th]].

[15] X. Dong, X. L. Qi, Z. Shangnan and Z. Yang, "Effective entropy of quantum fields coupled with gravity," JHEP **10**, 052 (2020) doi:10.1007/JHEP10(2020)052 [arXiv:2007.02987 [hep-th]].

[16] Tadashi Takayanagi, "Lectures in "The 15th Kavli Asian Winter School on Strings, Particles and Cosmology", lecture III and lecture IV.

[17] R. Blumenhagen and E. Plauschinn, "Introduction to conformal field theory: with applications to String theory," Lect. Notes Phys. **779**, 1-256 (2009) doi:10.1007/978-3-642-00450-6

[18] M. Rangamani and T. Takayanagi, "Holographic Entanglement Entropy," Lect. Notes Phys. **931**, pp.1-246 (2017) doi:10.1007/978-3-319-52573-0 [arXiv:1609.01287 [hep-th]].

[19] H. Z. Chen, R. C. Myers, D. Neuenfeld, I. A. Reyes and J. Sandor, "Quantum Extremal Islands Made Easy, Part I: Entanglement on the Brane," JHEP **10**, 166 (2020) doi:10.1007/JHEP10(2020)166 [arXiv:2006.04851 [hep-th]].

[20] A. Almheiri, R. Mahajan and J. E. Santos, "Entanglement islands in higher dimensions," SciPost Phys. **9**, no.1, 001 (2020) doi:10.21468/SciPostPhys.9.1.001 [arXiv:1911.09666 [hep-th]].

[21] N. Engelhardt and A. C. Wall, "Quantum Extremal Surfaces: Holographic Entanglement Entropy beyond the Classical Regime," JHEP **01**, 073 (2015) doi:10.1007/JHEP01(2015)073 [arXiv:1408.3203 [hep-th]].

[22] T. Anegawa and N. Iizuka, "Notes on islands in asymptotically flat 2d dilaton black holes," JHEP **07**, 036 (2020) doi:10.1007/JHEP07(2020)036 [arXiv:2004.01601 [hep-th]].

[23] R. Li, X. Wang and J. Wang, "Island may not save the information paradox of Liouville black holes," [arXiv:2105.03271 [hep-th]].

[24] H. Geng, A. Karch, C. Perez-Pardavila, S. Raju, L. Randall, M. Riojas and S. Shashi, "Information Transfer with a Gravitating Bath," SciPost Phys. **10**, no.5, 103 (2021) doi:10.21468/SciPostPhys.10.5.103 [arXiv:2012.04671 [hep-th]].

[25] D. Marolf and H. Maxfield, "Observations of Hawking radiation: the Page curve and baby universes," JHEP **04**, 272 (2021) doi:10.1007/JHEP04(2021)272 [arXiv:2010.06602 [hep-th]].

[26] D. Marolf and H. Maxfield, "The Page curve and baby universes," [arXiv:2105.12211 [hep-th]].